Momentum Conservation

For charges q_i at positions $\vec{r}_i$ with velocities $\dot{\vec{r}}_i$

$$\underbrace{\frac{d\vec{P}^{mech}}{dt}}_{\text{"mechanical" momentum of the charges}} = \sum_i \underbrace{\vec{F}_i}_{\text{force on charge } i} = \sum_i q_i\left(\vec{E}(\vec{r}_i) + \frac{1}{c}\vec{v}_i \times \vec{B}(\vec{r}_i)\right)$$

$$= \int_V d^3r \left[\rho\vec{E} + \frac{1}{c}\vec{J}\times\vec{B}\right]$$

$$\rho E + \frac{1}{c}\vec{J}\times\vec{B} = \frac{1}{4\pi}\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + \left(\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}\right]$$

Now $\frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{1}{c}\left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right) + \frac{1}{c}\left(\vec{E}\times\frac{\partial\vec{B}}{\partial t}\right)$ use $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$

$$= \frac{1}{c}\left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right) - \vec{E}\times(\vec{\nabla}\times\vec{E})$$

So $-\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\times\vec{B} = -\vec{E}\times(\vec{\nabla}\times\vec{E}) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$

Therefore

$$\rho E + \frac{1}{c}\vec{J}\times\vec{B} = \frac{1}{4\pi}\Big[\vec{E}(\vec{\nabla}\cdot\vec{E}) + \vec{B}\underbrace{(\vec{\nabla}\cdot\vec{B})}_{=0} - \vec{B}\times(\vec{\nabla}\times\vec{B}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})\Big]$$

Define <u>electromagnetic momentum density</u>

$$\boxed{\vec{\Pi} = \frac{1}{4\pi c}\vec{E}\times\vec{B} = \frac{1}{c^2}\vec{S}}$$ ($\vec{S}$ is Poynting vector)

then

$$\frac{d\vec{P}^{mech}}{dt} + \frac{d}{dt}\int_V d^3r\,\vec{\Pi} = \frac{1}{4\pi}\int_V d^3r \Big[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) + \vec{B}(\vec{\nabla}\cdot\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{B})\Big]$$

want to rewrite as a surface integral

ith component of integrand on right hand side is ($\vec{E}$ part only)
(sum over repeated indicies)

$$E_i \partial_j E_j - \varepsilon_{ijk} E_j \varepsilon_{klm} \partial_l E_m$$

$$= E_i \partial_j E_j - (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) E_j \partial_l E_m$$

$$= E_i \partial_j E_j - E_j \partial_i E_j + E_j \partial_j E_i$$

$$= \partial_j \left(E_i E_j - \tfrac{1}{2}\delta_{ij} E^2 \right)$$

Define <u>Maxwell's stress tensor</u>

$$\boxed{T_{ij} = \frac{1}{4\pi}\left[E_i E_j + B_i B_j - \tfrac{1}{2}\delta_{ij}(E^2 + B^2) \right]}$$

(note $T_{ij} = T_{ji}$
symmetric tensor

Then

$$\frac{d}{dt} P_i^{mech} + \frac{d}{dt}\int_V d^3r\, \Pi_i = \int_V d^3r\, \partial_j T_{ij}$$

$$\left(\partial_j T_{ij} = \frac{\partial T_{ij}}{\partial x_j} \right)$$

$$= \oint_S da\, T_{ij} \cdot \hat{n}_j$$

$$\frac{d}{dt}\vec{P}^{mech} + \frac{d}{dt}\int_V d^3r\, \vec{\Pi} = \oint_S da\, \overleftrightarrow{T} \cdot \hat{n}$$

- T_{ij} gives the flow of the i^{th} component of electromagnetic field momentum through an element of surface area $\perp$ ~~to~~ to direction $\hat{e}_j$

~~For static situations where $\frac{\partial \vec{\Pi}}{\partial t} = 0$, $\frac{d\vec{P}^{mech}}{dt} = \vec{F}^{tot} = \oint_S da\, \overleftrightarrow{T} \cdot \hat{n}$~~
~~gives electromagnetic force on the surface S~~

Note: $\frac{d\vec{P}^{mech}}{dt}$ is ~~also~~ equal to the total electromagnetic force on the volume V.

Hence we can write

$$\vec{F}_{EM} = \oint_S da\, \overleftrightarrow{T} \cdot \hat{n} - \frac{d}{dt} \int_V d^3r\, \vec{\Pi}$$

for static situations, the 2nd term vanishes and

$$\vec{F}_{EM} = \oint_S da\, \overleftrightarrow{T} \cdot \hat{n}$$

$\overleftrightarrow{T}_{ij}$ is i^{th} component of static force on unit area with normal $\hat{e}_j$

this is origin of the term "stress" tensor.

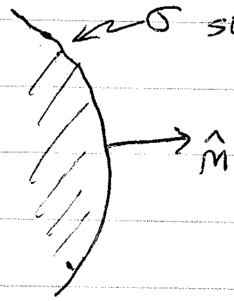
$\overleftrightarrow{T}$ is like the stress tensor of an elastic medium,

T_{xx}, T_{yy}, T_{zz} are like pressure.

Off diagonal elements are like shear stresses

Force on a conductor surface

σ surface charge on conductor



net force on surface per unit area is

$$\vec{f} = \overleftrightarrow{T}^{\,above} \cdot \hat{n} - \overleftrightarrow{T}^{\,below} \cdot \hat{n}$$

$\uparrow = 0$ as $\vec{E} = 0$ inside conductor

$$\vec{f} = \frac{1}{4\pi}\left[\vec{E}\,(\vec{E}\cdot\hat{n}) - \tfrac{1}{2}\hat{n}E^2\right]$$

for conducting surface

$\hat{n}\cdot\vec{E}^{\,above} = 4\pi\sigma$ (since $\vec{E}$ below $= 0$)

and tangential component $\vec{E} = 0$

$$\Rightarrow \vec{E} = 4\pi\sigma\,\hat{n}$$

So $\vec{f} = \frac{1}{4\pi}\left[(4\pi\sigma\,\hat{n})(4\pi\sigma) - \tfrac{1}{2}\hat{n}\,(4\pi\sigma)^2\right]$

$\cancel{\vec{f} = 2\pi\sigma^2\hat{n}}$

$$\vec{f} = \frac{\hat{n}}{4\pi}\left[(4\pi\sigma)^2 - \tfrac{1}{2}(4\pi\sigma)^2\right] = 2\pi\sigma^2\,\hat{n}$$

force per unit area: $\boxed{\vec{f} = 2\pi\sigma^2\hat{n} = \tfrac{1}{2}\sigma\vec{E}}$

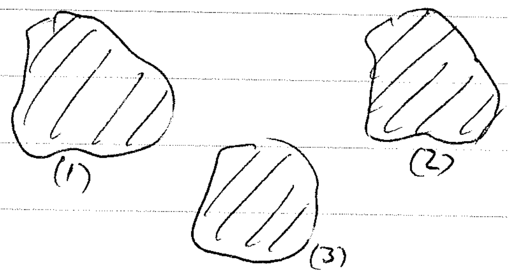
$\uparrow$ Note factor ½. Naively one might have thought $\vec{f} = \sigma\vec{E}$. But need to exclude self field of charge on surface from acting on itself. See also Jackson pg 42 for another approach.

$\vec{f} = \sigma\bar{E}_{ave}$

where $\vec{E}_{ave} = \frac{1}{2}(\vec{E}_{above} + \vec{E}_{below})$ is average field at surface averaging over above + below

Capacitance

Consider a set of conductors with potential $\phi(\vec{r}) = V_i$ fixed on conductor i



(also need condition on $\phi(\vec{r}) \to \infty$ if system is not enclosed)

From uniquess theorem we know that specifying the V_i on each conductor is enough to determine the potential $\phi(\vec{r})$ everywhere. We can write this potential in the following form.

Let $\phi^{(i)}(\vec{r})$ be the solution to the boundary value problem

$$\nabla^2 \phi^{(i)}(r) = 0 \quad \text{and} \quad \phi^{(i)}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \text{ on surface of conductor } (i) \\ 0 & \text{if } \vec{r} \text{ on surface of any other conductor } (j), \; j \neq i \end{cases}$$

Then by superposition

$$\phi(\vec{r}) = \sum_i V_i \, \phi^{(i)}(\vec{r})$$

is solution to the problem $\nabla^2 \phi = 0$ and $\phi(\vec{r}) = V_i$ for $\vec{r}$ on surface of conductor (i)

The surface charge density at $\vec{r}$ on surface of conductor (i) is

$$\sigma^{(i)}(\vec{r}) = \frac{-1}{4\pi} \frac{\partial \phi(\vec{r})}{\partial n} = -\frac{1}{4\pi} \sum_j V_j \frac{\partial \phi^{(j)}(\vec{r})}{\partial n}$$

where $\dfrac{\partial \phi}{\partial n} = (\vec{\nabla} \phi) \cdot \hat{n}$ is the derivative normal to the surface at point $\vec{r}$.

The total charge on conductor (i) is

$$Q_i = \int_{S_i} da\, \sigma^{(i)}(\vec{r}) = -\frac{1}{4\pi}\sum_j V_j \int_{S_i} da\, \frac{\partial \phi^{(j)}}{\partial n}$$

surface of conductor (i)

Define $C_{ij} \equiv -\frac{1}{4\pi}\int_{S_i} da\, \frac{\partial \phi^{(j)}}{\partial n}$

the C_{ij} depend only on the geometry of the conductors

Then we have

$$\boxed{Q_i = \sum_j C_{ij} V_j}$$

C_{ij} is the capacitance matrix

The charge on conductor (i) is a linear function of the potentials V_j on the conductors (j)

Since we know that specifying the Q_i that is on each conductor will uniquely determine $\phi(\vec{r})$ and hence the potential V_i on each conductor, the capacitance matrix is invertable

$$V_i = \sum_j [C^{-1}]_{ij} Q_j$$

The electrostatic energy of the conductors is then

$$\mathcal{E} = \frac{1}{2}\int d^3r\, \rho\, \phi = \frac{1}{2}\sum_i Q_i V_i = \frac{1}{2}\sum_{i,j} C_{ij} V_i V_j = \frac{1}{2}\sum_{ij} C^{-1}_{ij} Q_i Q_j$$

Common to define capacitance of two conductors by

$$C = \frac{Q}{V_1 - V_2}$$ when conductor (1) has charge Q
conductor (2) has charge $-Q$
$V_1 - V_2$ is potential difference between the two conductors.
all other conductors fixed at $V_i = 0$

We can determine C in terms of the elements of the matrix C_{ij}

$$\left.\begin{aligned} Q &= C_{11}V_1 + C_{12}V_2 \\ -Q &= C_{21}V_1 + C_{22}V_2 \end{aligned}\right\} \Rightarrow V_2 = -\left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)V_1$$

$$\Rightarrow Q = \left[C_{11} - C_{12}\left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)\right]V_1$$

$$V_1 - V_2 = \left[1 + \left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)\right]V_1$$

$$C = \frac{Q}{V_1 - V_2} = \frac{C_{11} - C_{12}\left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)}{1 + \left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)}$$

$$\boxed{C = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11} + C_{12} + C_{21} + C_{22}}}$$

Capacitance can also be defined when the space between the conductors is filled with a dielectric ϵ. In this case, if Q_i is the free charge, then Q_i/ϵ is the effective total charge to use in computing ϕ.

$$\Rightarrow \frac{Q_i}{\epsilon} = \sum_j C_{ij}^{(0)} V_j$$ where $C_{ij}^{(0)}$ are capacitances appropriate to a vacuum between the conductors

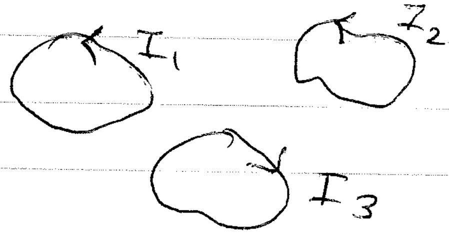
$$\Rightarrow Q_i = \sum_j \epsilon\, C_{ij}^{(0)} V_j$$

$$= \sum_j C_{ij} V_j \quad \text{where } C_{ij} = \epsilon\, C_{ij}^{(0)}$$

the capacitance is increased by a factor the dielectric constant ϵ.

Inductance

Consider a set of current carrying loops C_i with currents I_i



In Coulomb gauge, we can write the magnetic vector potential $\vec{A}$ from these current loops as

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \sum_i \frac{I_i}{c} \oint_{C_i} \frac{d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

integrate over loop C_i
integration variable is $\vec{r}'$

The magnetic flux through loop i is

$$\Phi_i = \int_{S_i} da\, \hat{n}\cdot\vec{B} = \int_{S_i} da\, \hat{n}\cdot\vec{\nabla}\times\vec{A} = \oint_{C_i} d\vec{l}\cdot\vec{A}$$

surface bounded by loop C_i

$$\Phi_i = \sum_j \frac{I_j}{c} \oint_{C_i}\oint_{C_j} \frac{d\vec{l}\cdot d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

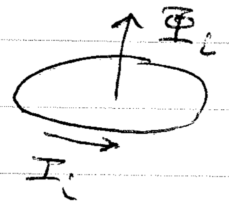
pure geometrical quantity

$$\boxed{\Phi_i \equiv c\sum_j M_{ij} I_j}$$ where $M_{ij} = \oint_i \oint_j \frac{d\vec{l}_i\cdot d\vec{l}_j}{c^2|\vec{r}-\vec{r}'|}$

is the <u>mutual inductance</u> of loops (i) and (j). $M_{ij} = M_{ji}$

$L_i \equiv M_{ii}$ is <u>self-inductance</u> of loop (i)

The sign convention in the above is that Φ_i is computed in direction given by right hand rule, according to the direction taken for current in loop (i)



Magnetostatic energy

$$\mathcal{E} = \frac{1}{2c}\int d^3r \, \vec{j}\cdot\vec{A} = \frac{1}{2c}\sum_i \oint_{C_i} d\vec{l}\cdot\vec{A} \; I_i$$

$$= \frac{1}{2c}\sum_i \Phi_i I_i$$

$$\mathcal{E} = \frac{1}{2}\sum_{i,j} M_{ij} I_i I_j$$