

Electromagnetic waves in a vacuum

No sources $\vec{j} = 0$, $\rho = 0$

$$\begin{array}{ll} 1) \quad \vec{\nabla} \cdot \vec{E} = 0 & 3) \quad \vec{\nabla} \cdot \vec{B} = 0 \\ 2) \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} & 4) \quad \vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \end{array}$$

$$\nabla \times (2) \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B})$$

0'' by (1)

$$-\nabla^2 \vec{E} = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) = -\frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right)$$

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Similarly

$$\nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

} wave equation
wave speed is c .

Note: in MKS units, above wave equation looks like

$$\nabla^2 \vec{E} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

It was noticed that the speed of electromagnetic wave,

$$\frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3 \times 10^8 \text{ m/s}$$

was the same as the speed of

light! This observation was a key element in showing that light was in fact electromagnetic waves

Harmonic

Plane waves

$$\begin{aligned}\vec{E}(\vec{r}, t) &= \text{Re} \left[\vec{E}_k e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right] \\ \vec{B}(\vec{r}, t) &= \text{Re} \left[\vec{B}_k e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right]\end{aligned} \quad \left. \vphantom{\begin{aligned}\vec{E}(\vec{r}, t) \\ \vec{B}(\vec{r}, t)\end{aligned}} \right\} \text{complex exponential form}$$

\vec{k} is wave vector

ω is angular frequency

$\nu = \omega/2\pi$ is frequency

$T = 1/\nu$ is period

$\lambda = \frac{2\pi}{|\vec{k}|}$ is wavelength

$\left. \begin{array}{l} |\vec{E}_k| \\ |\vec{B}_k| \end{array} \right\}$ is amplitude

$$\vec{E}(\vec{r} + \lambda \hat{k}, t) = \vec{E}(\vec{r}, t)$$

periodic in space with period λ

$$\vec{E}(\vec{r}, t + T) = \vec{E}(\vec{r}, t)$$

periodic in time with period T

"plane wave" $\Rightarrow \vec{E}(\vec{r}, t)$ is constant in space on planes with normal $\hat{m} \parallel \vec{k}$.

properties of EM plane waves

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} = 0 &\Rightarrow \text{Re} \left[\vec{E}_k \cdot \vec{\nabla} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right] \\ &= \text{Re} \left[i \vec{E}_k \cdot \vec{k} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right] = 0 \\ &\Rightarrow \vec{E}_k \cdot \vec{k} = 0\end{aligned}$$

amplitude is orthogonal to \vec{k}

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B}_k \cdot \vec{k} = 0$$

amplitude orthogonal to \vec{k}

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \operatorname{Re} \left[\vec{\nabla} \times \vec{B}_k e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right] = \operatorname{Re} \left[\frac{1}{c} \vec{E}_k \frac{\partial}{\partial t} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right]$$

$$\Rightarrow \operatorname{Re} \left[-\vec{B}_k \times \vec{\nabla} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right] = \operatorname{Re} \left[-\frac{i\omega}{c} \vec{E}_k e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right]$$

$$\Rightarrow \operatorname{Re} \left[i\vec{k} \times \vec{B}_k e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right] = \operatorname{Re} \left[-\frac{i\omega}{c} \vec{E}_k e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right]$$

$$\Rightarrow \vec{k} \times \vec{B}_k = -\frac{\omega}{c} \vec{E}_k$$

$$\vec{k} \times \vec{k} \times \vec{B}_k = -k^2 \vec{B}_k = -\frac{\omega}{c} \vec{k} \times \vec{E}_k$$

$$\underline{\vec{B}_k = \frac{\omega}{ck^2} \vec{k} \times \vec{E}_k}$$

Finally

$$\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

$$\Rightarrow \operatorname{Re} \left[\vec{E}_k \nabla^2 e^{i(\vec{k} \cdot \vec{r} - \omega t)} - \frac{\omega^2}{c^2} \frac{\partial^2}{\partial t^2} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right] = 0$$

$$\Rightarrow \operatorname{Re} \left[\vec{E}_k (-k^2) e^{i(\vec{k} \cdot \vec{r} - \omega t)} + \frac{\omega^2}{c^2} \vec{E}_k e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right] = 0$$

$$\Rightarrow k^2 = \frac{\omega^2}{c^2}$$

$$\boxed{\omega = \pm kc} \quad \underline{\text{dispersion relation}}$$

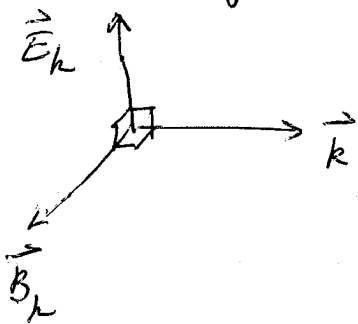
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$$\vec{B}_k = \hat{k} \times \vec{E}_k$$

$$\hat{k} = \frac{\vec{k}}{|\vec{k}|}$$

$$\Rightarrow |\vec{B}| = |\vec{E}|$$

Summary



$$\left. \begin{aligned} \vec{E}_k &\perp \vec{k} \\ \vec{B}_k &\perp \vec{k} \end{aligned} \right\} \text{"transverse" polarization}$$

$$\vec{B}_k = \hat{k} \times \vec{E}_k$$

$$\omega^2 = c^2 k^2$$

$|\vec{B}_k| = |\vec{E}_k| \Rightarrow$ Lorentz force from plane EM wave on charge q is

$$q \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right)$$

magnetic force is smaller factor $\left(\frac{v}{c}\right)$ as compared to electric force - can usually be ignored

Most general solution is a linear superposition of the above ^{harmonic} plane waves

$$\vec{E}(\vec{r}, t) = \int \frac{d^3k}{(2\pi)^3} \vec{E}_k e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

Fourier transform

$$\vec{E}(\vec{r}, t) \text{ is real} \Rightarrow \vec{E}_k^* = \vec{E}_{-k}$$

For dispersion relation $\omega^2 = c^2 k^2$ we can write $\vec{k} \cdot \vec{r} - \omega t = \vec{k}_0 \cdot (\vec{r} - \vec{v} t)$

where $\vec{v} = c \hat{k}$ is velocity of wave. If we only combine waves traveling in same direction \hat{k} , then

$$\vec{E}(\vec{r}, t) = \int \frac{d^3k}{(2\pi)^3} \vec{E}_k e^{i\vec{k}_0 \cdot (\vec{r} - \vec{v} t)} = \vec{E}(\vec{r} - \vec{v} t, 0)$$

The general ^{plane wave} solution of wave equation always has this property

$$\vec{E}(\vec{r}, t) = \vec{E}(\vec{r} - \vec{v} t, 0)$$

If know \vec{E} at $t=0$, then know \vec{E} at all times t

Energy & momentum in EM wave

$$\begin{aligned}\vec{E} &= \text{Re} [\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}] = \vec{E}_k \cos(\vec{k}\cdot\vec{r}-\omega t) \\ \vec{B} &= \text{Re} [\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}] = \hat{k} \times \vec{E}_k \cos(\vec{k}\cdot\vec{r}-\omega t)\end{aligned} \left. \vphantom{\begin{aligned}\vec{E} \\ \vec{B}\end{aligned}} \right\} \begin{array}{l} \text{for} \\ \text{real} \\ \vec{E}_k \end{array}$$

energy density $u = \frac{1}{8\pi} (E^2 + B^2)$

$$= \frac{1}{8\pi} [E_k^2 + E_k^2] \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$= \frac{1}{4\pi} E_k^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

Poynting vector

energy current

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$

$$= \frac{c}{4\pi} [\vec{E}_k \times (\hat{k} \times \vec{E}_k)] \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$= \frac{c}{4\pi} \hat{k} E_k^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$\vec{S} = c u \hat{k}$$

momentum density $\vec{\Pi} = \frac{1}{c^2} \vec{S} = \frac{u}{c} \hat{k}$

$$u = c |\vec{\Pi}| \quad - \text{energy momentum relation of photons!}$$

For visible light $\lambda \sim 5 \times 10^{-7} \text{ m} \sim 5000 \text{ \AA}$

$$T = \frac{\lambda}{c} = 1.6 \times 10^{-15} \text{ sec}$$

most classical measurements on microscopic scales $t \gg T$, $l \gg \lambda$

measure average quantities

$$\langle u \rangle \equiv \frac{1}{T} \int_0^T dt u = \frac{1}{8\pi} E_k^2 \quad \text{as } \langle \cos^2 \theta \rangle = \frac{1}{2}$$

$$\langle \vec{S} \rangle = c \langle u \rangle \hat{k}$$

$$\langle \vec{\Pi} \rangle = \frac{1}{c} \langle u \rangle \hat{k}$$

intensity = average power per area transported by wave through surface with normal \hat{n}

$$I = \langle \vec{S} \rangle \cdot \hat{n}$$

Electromagnetic waves in matter

Macroscopic Maxwell equations with no sources
("free" charge and current vanishes)

$$\begin{aligned}\vec{\nabla} \cdot \vec{D} &= 0 & \vec{\nabla} \times \vec{H} &= \frac{1}{c} \frac{\partial \vec{D}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{c \partial t}\end{aligned}$$

linear materials

$$\begin{aligned}\vec{B} &= \mu \vec{H} \\ \vec{D} &= \epsilon \vec{E}\end{aligned}$$

if μ and ϵ were simply constants then the above would become

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= 0 & \vec{\nabla} \times \vec{B} &= \frac{\mu \epsilon}{c} \frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}\end{aligned}$$

Then

$$\begin{aligned}\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= -\nabla^2 \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) \\ &= -\frac{1}{c} \frac{\partial}{\partial t} \left(\frac{\mu \epsilon}{c} \frac{\partial \vec{E}}{\partial t} \right) \\ &= -\frac{\mu \epsilon}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}\end{aligned}$$

wave equation with wave speed $\frac{c}{\sqrt{\mu \epsilon}} < c$

This would be very much as for waves in a vacuum, except for the following minus

changes:

$$\omega^2 = \frac{c^2 k^2}{\mu \epsilon}$$

dispersion relation
changed by constant
factor

$$\begin{aligned}\vec{E}_k &\perp \vec{k} \\ \vec{B}_k &\perp \vec{k}\end{aligned}$$

$$i \vec{k} \times \vec{E}_k = i \frac{\omega}{c} \vec{B}_k$$

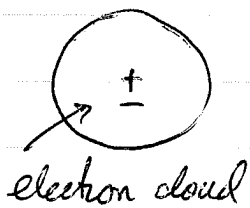
$$\frac{c |\vec{k}|}{\omega} \hat{k} \times \vec{E}_k = \vec{B}_k$$

$$\Rightarrow \sqrt{\mu \epsilon} \hat{k} \times \vec{E}_k = \vec{B}_k \quad |\vec{B}_k| > |\vec{E}_k|$$

wave speed $v = \frac{c}{\sqrt{\mu \epsilon}} < c$

In general however, things are much more complicated
because ϵ cannot be viewed as a constant
when considering time varying behavior!

Time dependent polarizability of an atom



If displace center of electron cloud by a distance \vec{r} , there is a restoring force $\vec{F}_{\text{rest}} = -\frac{e^2 \vec{r}}{4\pi R^3} \equiv -m\omega_0^2 \vec{r}$

↑
electron mass
↑
resonant frequency

Also, in general there will be a damping force

$$\vec{F}_{\text{damp}} = -m\gamma \frac{d\vec{r}}{dt}$$

due to transfer of energy from atom to other degrees of freedom.

In an external electric field $\vec{E}(t)$, the equation of motion for electron cloud is

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F}_{\text{tot}} = -e \vec{E}(t) - m\omega_0^2 \vec{r} - m\gamma \frac{d\vec{r}}{dt}$$

$$\frac{d^2 \vec{r}}{dt^2} + \gamma \frac{d\vec{r}}{dt} + \omega_0^2 \vec{r} = -\frac{e \vec{E}(t)}{m}$$

assuming \vec{E} is spatially constant over atomic distances

For harmonic oscillation $\vec{E}(t) = \vec{E}_0 e^{-i\omega t}$

Assume solution $\vec{r}(t) = \vec{r}_0 e^{-i\omega t}$

(in the end, we will take the real parts)

Substitute into equation of motion

$$-\omega^2 \vec{r}_0 - i\omega \gamma \vec{r}_0 + \omega_0^2 \vec{r}_0 = -\frac{e \vec{E}_0}{m}$$

$$\vec{r}_0 = \frac{-e}{m(\omega_0^2 - \omega^2 - i\omega\gamma)} \vec{E}_0$$

polarization

$$\vec{\Phi} = -e\vec{r} = \vec{\Phi}_0 e^{-i\omega t}$$

$$\vec{\Phi}_0 = \frac{e^2}{m} \frac{1}{(\omega_0^2 - \omega^2 - i\omega\gamma)} \vec{E}_0 = \alpha(\omega) \vec{E}_0$$

$$\alpha(\omega) = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} \quad \text{freq dependent polarizability}$$

Since α is complex the polarization does not in general oscillate in phase with \vec{E} .

If $\alpha(\omega) = |\alpha| e^{i\delta}$ δ is phase of complex α

$$\vec{\Phi}(t) = \alpha(\omega) \vec{E}(t) = |\alpha| e^{i\delta} \vec{E}_0 e^{-i\omega t} = |\alpha| \vec{E}_0 e^{-i(\omega t - \delta)}$$

↑
phase shifted by δ

For a general electric field

$$\vec{E}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \vec{E}_\omega e^{-i\omega t}$$

$$\vec{\Phi}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Phi_\omega e^{-i\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) \vec{E}_\omega e^{-i\omega t}$$

$\vec{E}_\omega^* = \vec{E}_{-\omega}$

Substitute in $\vec{E}_\omega = \int_{-\infty}^{\infty} dt' E(t') e^{i\omega t'}$ to get

$$\vec{\Phi}(t) = \int_{-\infty}^{\infty} dt' \vec{E}(t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) e^{-i\omega(t-t')}$$

$$\vec{\Phi}(t) = \int_{-\infty}^{\infty} dt' \vec{E}(t') \tilde{\alpha}(t-t')$$

↑ Fourier transf of $\alpha(\omega)$

$\vec{\Phi}$ at time t is due to \vec{E} at all times t'
non local in time