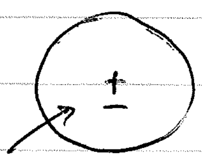
# Time dependent polarizeability of an atom



electron cloud

If displace center of electron cloud by a distance $\vec{r}$, there is a restoring force $\vec{F}_{rest} = -\frac{e^2 \vec{r}}{4\pi R^3} \equiv -m\omega_0^2 \vec{r}$

electron mass  resonant frequency

Also, in general there will be a damping force

$$\vec{F}_{damp} = -m\gamma \frac{d\vec{r}}{dt}$$

due to transfer of energy from atom to other degrees of freedom.

In an external electric field $\vec{E}(t)$, the equation of motion for electron cloud is

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{tot} = -e\vec{E}(t) - m\omega_0^2\vec{r} - m\gamma\frac{d\vec{r}}{dt}$$

$$\ddot{\vec{r}} + \gamma\dot{\vec{r}} + \omega_0^2\vec{r} = -\frac{e\vec{E}(t)}{m}$$

assume $\vec{E}$ is spatially constant over atomic distances

For harmonic oscillation $\vec{E}(t) = \vec{E}_0 e^{-i\omega t}$

Assume solution $\vec{r}(t) = \vec{r}_0 e^{-i\omega t}$

(in the end, we will take the real parts)

Substitute into equation of motion

$$-\omega^2\vec{r}_0 - i\omega\gamma_0\vec{r}_0 + \omega_0^2\vec{r}_0 = -\frac{e\vec{E}_0}{m}$$

$$\vec{r}_0 = \frac{-e}{m(\omega_0^2 - \omega^2 - i\omega\gamma)}\vec{E}_0$$

polarization

$$\vec{p} = -e\vec{r} = \vec{p}_0 e^{-i\omega t}$$

$$\vec{p}_0 = \frac{e^2}{m} \frac{1}{(\omega_0^2 - \omega^2 - i\omega\gamma)} \vec{E}_0 = \alpha(\omega) E_0$$

$$\alpha(\omega) = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$ freq dependent polarizability

Since $\alpha$ is complex the polarization does not in general oscillate in phase with $\vec{E}$.

If $\alpha(\omega) = |\alpha| e^{i\delta}$ $\delta$ is phase of complex $\alpha$

$$\vec{p}(t) = \alpha(\omega)\vec{E}(t) = |\alpha| e^{i\delta} \vec{E}_0 e^{-i\omega t} = |\alpha| \vec{E}_0 e^{-i(\omega t - \delta)}$$

phase shifted by $\delta$

For a general electric field

$$\vec{E}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \vec{E}_\omega e^{-i\omega t}$$

$$\vec{E}_\omega^* = \vec{E}_{-\omega}$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} p_\omega e^{-i\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) \vec{E}_\omega e^{-i\omega t}$$

Substitute in $\vec{E}_\omega = \int_{-\infty}^{\infty} dt' \, E(t') e^{i\omega t'}$ to get

$$\vec{p}(t) = \int_{-\infty}^{\infty} dt' \, \vec{E}(t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) e^{-i\omega(t-t')}$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} dt' \, \vec{E}(t') \, \tilde{\alpha}(t-t')$$

Fourier transf of $\alpha(\omega)$

$\vec{p}$ at time $t$ is due to $\vec{E}$ at all times $t'$
non local in time

$\tilde{\alpha}(t)$ is the response to $\vec{E}(t) = \delta(t)$

For our simple model

$$\tilde{\alpha}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \frac{e^2}{m}\, \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

do by contour integration

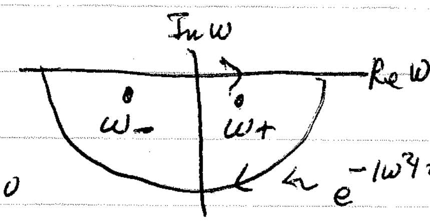
$$\frac{1}{\omega^2 + i\gamma\omega - \omega_0^2} = \frac{1}{(\omega - \omega_+)(\omega - \omega_-)}$$

$$\omega_\pm = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \gamma^2/4} = -\frac{i\gamma}{2} \pm \bar{\omega}$$

poles at $\omega_\pm$ are in lower half complex plane.

for $t > 0$, close contour in lower half plane

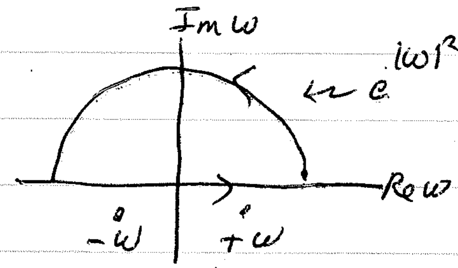
$t > 0$




$\leftarrow e^{-|\omega|^2 t}$ here vanishes for $t > 0$

contour encloses poles & get contribution

for $t < 0$, close contour in upper half plane




$\leftarrow e^{|\omega|^2 t}$ vanishes for $t < 0$

contour encloses no poles $\Rightarrow$ integral vanishes

$\tilde{\alpha}(t) = 0$ for $t < 0$

<u>causal response</u>! No polarization until electric field turns on

For $t > 0$

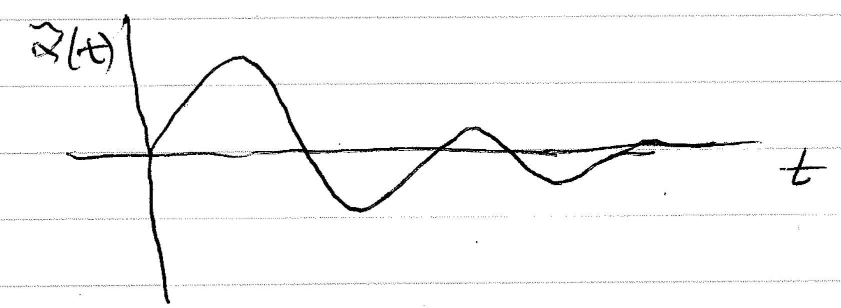
$$\tilde{\alpha}(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{e^2}{m} \frac{(-1)}{(\omega-\omega_+)(\omega-\omega_-)}$$

$$= (-2\pi i) \frac{e^2}{m} \frac{(-1)}{2\pi} \left[ \frac{e^{-i\omega_+ t}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_- t}}{\omega_- - \omega_+} \right]$$

from Residue theorem

$$= \frac{i e^2}{m} \left[ \frac{e^{-\gamma t/2} e^{-i\bar{\omega} t}}{2\bar{\omega}} - \frac{e^{-\gamma t/2} e^{i\bar{\omega} t}}{2\bar{\omega}} \right]$$

$$\tilde{\alpha}(t) = \begin{cases} \frac{e^2}{m} \frac{e^{-\gamma t/2}}{\bar{\omega}} \sin(\bar{\omega} t) & t > 0 \\ 0 & t < 0 \end{cases}$$




dayed oscillation

Polarization density $\vec{P}_\omega = 4\pi \chi(\omega) \vec{E}_\omega$ for harmonic oscillation

$\chi(\omega) \approx n\, \alpha(\omega)$ for dilute system

atom density

can use Clausius - Mossotti correction for denser materials

$\Rightarrow \vec{D}_\omega = \varepsilon(\omega) \vec{E}_\omega$ $\qquad \varepsilon(\omega) = 1 + 4\pi \chi(\omega)$

freq dependent

→ as with $\vec{J}$ and $\vec{E}$, relation between $\vec{D}$ and $\vec{E}$ is <u>non-local</u> in time

$$\vec{D}(t) \neq \varepsilon \vec{E}(t)$$

rather

$$\vec{D}(t) = \int_{-\infty}^{\infty} dt' \, \vec{E}(t') \, \tilde{\varepsilon}(t-t')$$

Fourier transf of $\varepsilon(\omega)$

Ampere's law is

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

becomes $$\frac{1}{\mu} \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\int_{-\infty}^{\infty} dt' \, \vec{E}(t') \frac{d}{dt} \tilde{\varepsilon}(t-t')$$

integro-differential-equation!

Maxwells equations only look simple when expressed in terms of Fourier transforms

$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{B}(\vec{r},t) = \vec{B}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{D}(\vec{r},t) = \vec{D}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{H}(\vec{r},t) = \vec{H}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Maxwell's Equ for <u>source free</u> system $\rho = \vec{J} = 0$

$$\vec{\nabla}\cdot\vec{D} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} = \frac{-\partial \vec{B}}{c\,\partial t}, \quad \vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{c\,\partial t}$$

assume $\mu$ is true constant - not freq dependent

dielectric response is $\vec D_\omega = \varepsilon(\omega)\vec E_\omega$

Then for the Fourier amplitudes of the fields, Maxwell's Equations become

transverse polarized

1) $i\vec k \cdot \vec D_\omega = i\,\varepsilon(\omega)\,\vec k \cdot \vec E_\omega = 0 \Rightarrow \boxed{\vec k \perp \vec E_\omega}$ (unless $\varepsilon(\omega) = 0$)

2) $i\vec k \cdot \vec B_\omega = 0 \Rightarrow \boxed{\vec k \perp \vec B_\omega}$

3) $i\vec k \times \vec E_\omega = i\frac{\omega}{c}\vec B_\omega$

4) $i\vec k \times \vec H_\omega = -i\frac{\omega}{c}\vec D_\omega \Rightarrow \frac{i\vec k}{\mu} \times \vec B_\omega = -\frac{i\omega}{c}\varepsilon(\omega)\vec E_\omega$

$$\vec k \times (3) = i\vec k \times (\vec k \times \vec E_\omega) = i\frac{\omega}{c}\vec k \times \vec B_\omega$$

$$\Rightarrow -ik^2\vec E_\omega = -\frac{i\omega^2}{c^2}\varepsilon(\omega)\mu\,\vec E_\omega \quad \text{using (4)}$$

$$\boxed{k^2 = \frac{\omega^2}{c^2}\varepsilon(\omega)\mu}$$

dispersion relation

$\cancel{\vec B_\omega = \frac{\vec k \times \vec E_\omega}{\omega} = \sqrt{\varepsilon(\omega)\mu}\,\hat k \times \vec E_\omega}$

**Note:** $\frac{\omega}{|k|} = \frac{c}{\sqrt{\varepsilon(\omega)\mu}}$ varies with $\omega$.

There is not a single phase velocity.

$\Rightarrow \vec E$ is not in general a solution of a wave equation - different frequencies travel with different speeds

Since $\varepsilon(\omega)$ is complex $\quad \varepsilon(\omega) = \varepsilon_1(\omega) + i\,\varepsilon_2(\omega)$

$\Rightarrow$ wave vector also complex $\quad$ For $\vec{k} = k\hat{z}$

$$k = k_1 + i k_2 = \pm \frac{\omega}{c}\sqrt{\mu}\,\sqrt{\varepsilon_1 + i\varepsilon_2}$$

$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{E}_\omega e^{i[(k_1+ik_2)z - \omega t]}$$
$$= \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$$

$k_1$ determines the oscillation of the wave

$k_2$ determines the decay or attenuation of the wave as it propagates into the material

phase velocity $\quad v_p = \dfrac{\omega}{k_1}$

index of refraction $\quad n = \dfrac{c}{v_p} = \dfrac{c k_1}{\omega}$

group velocity $v_g$ $\quad \dfrac{1}{v_g} = \dfrac{dk_1}{d\omega}$

Magnetic field : $\quad \vec{B}_\omega = \dfrac{c\vec{k}}{\omega} \times \vec{E}_\omega$

for $\vec{k} = k\hat{z}$, $\quad \vec{B}_\omega = \dfrac{c(k_1 + ik_2)}{\omega}\,\hat{z} \times \vec{E}_\omega$

if $\quad k_1 + ik_2 = \sqrt{k_1^2 + k_2^2}\; e^{i\delta} \qquad \delta = \arctan\left(\dfrac{k_2}{k_1}\right)$
$\quad = |k|\,e^{i\delta}$

$$\vec{B}_\omega = \frac{c|k|}{\omega}\,\hat{z} \times \vec{E}_\omega\, e^{i\delta}$$

phase shift

$$\vec{B}(\vec{r},t) = \frac{c|k|}{\omega} (\hat{z}\times\vec{E}_\omega)\, e^{-k_2 z}\, e^{i(k_1 z - \omega t + \delta)}$$

<u>Physical fields</u> - take real parts

$$\vec{E}(\vec{r},t) = \vec{E}_\omega\, e^{-k_2 z} \cos(k_1 z - \omega t)$$

$$\vec{B}(\vec{r},t) = (\hat{z}\times\vec{E}_\omega)\frac{c|k|}{\omega}\, e^{-k_2 z}\cos(k_1 z - \omega t + \delta)$$

<u>Conclusions</u>

1) $\vec{E}$ and $\vec{B}$ $\perp$ $\vec{k}$ transverse polarized
2) $\vec{E} \perp \vec{B}$
3) amplitude ratio $\frac{|\vec{B}|}{|\vec{E}|} = \frac{c|k|}{\omega} = \sqrt{|\varepsilon(\omega)|\mu}$
4) $\vec{B}$ is shifted in phase with respect to $\vec{E}$ by phase shift $\delta = \arctan(k_2/k_1)$
5) waves decay as they propagate $e^{-k_2 z}$

} consequence of complex $\varepsilon(\omega)$

If $\varepsilon_2 = 0$, ie $\varepsilon(\omega)$ is real, <u>and</u> if $\varepsilon > 0$, then $k_2 = 0 \Rightarrow$ no decay, no phase shift

consequences of frequency dependence of $\varepsilon(\omega)$ {

6) $\vec{E}(t)$ and $\vec{D}(t)$ non locally related in time
7) waves of different $\omega$ travel with different $v_p = \omega/k_1$
8) <u>dispersion</u> - wave pulses do <u>not</u> travel with $v_p$ and they spread as the propagate

pulses travel with group velocity $v_g = \frac{d\omega}{dk}$ (see Quantum Mechanics discussion)

$v_g < v_p$ "normal dispersion"

$v_g > v_p$ "anomalous dispersion"

index of refraction

$$\frac{1}{v_g} = \frac{dk_1}{d\omega} = \frac{d}{d\omega}\left[\frac{\omega}{c}n\right]$$

$$\frac{1}{v_g} = \frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega} = \frac{1}{v_p} + \frac{\omega}{c}\frac{dn}{d\omega}$$

$$v_g = \frac{v_p}{1 + \frac{v_p}{c}\,\omega\frac{dn}{d\omega}}$$

$$\Rightarrow \text{ when } \begin{cases} \frac{dn}{d\omega} > 0, & v_g < v_p \quad \text{normal dispersion} \\ \frac{dn}{d\omega} < 0, & v_g > v_p \quad \text{anomalous dispersion} \end{cases}$$

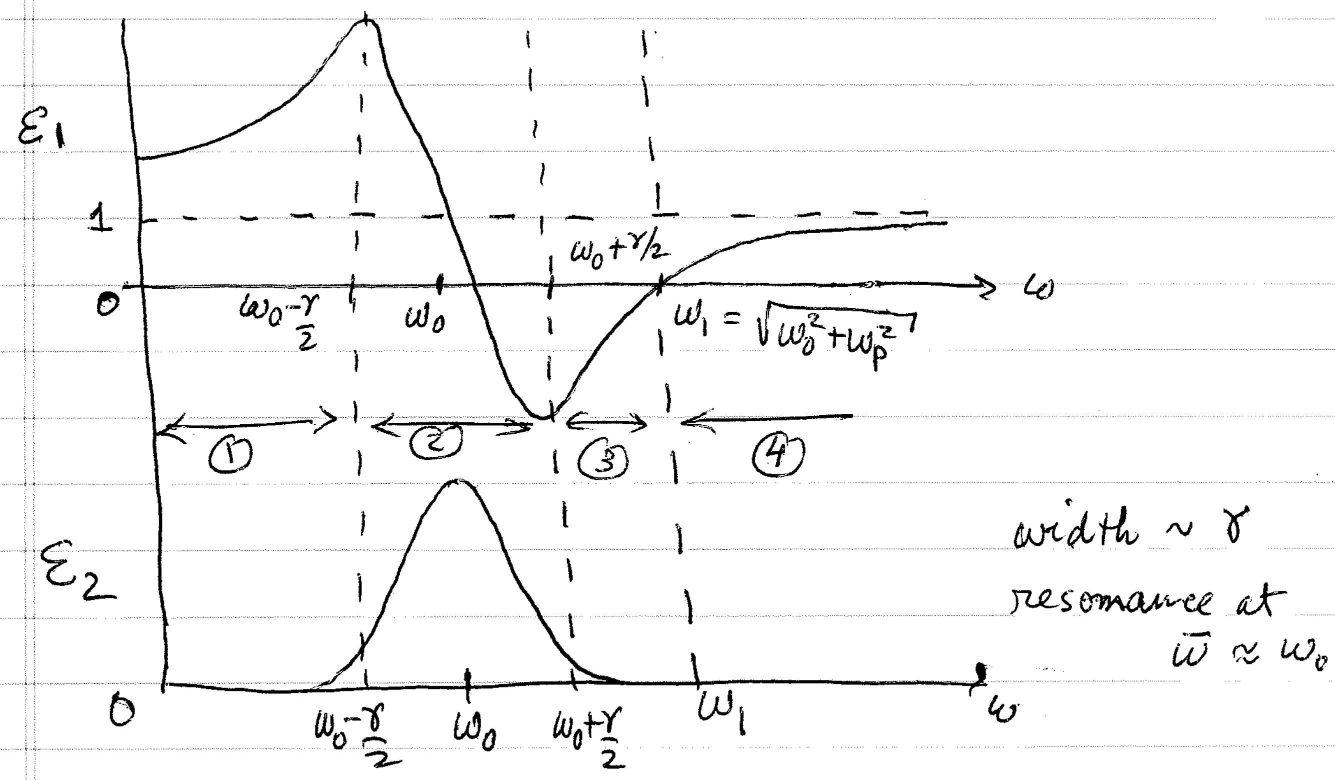
For our simple model: $\varepsilon = 1 + 4\pi\chi \simeq 1 + 4\pi M\alpha$

$$\varepsilon(\omega) = 1 + \frac{4\pi M e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

$$\varepsilon_1 = 1 + \frac{4\pi M e^2}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}$$

$$\varepsilon_2 = \frac{4\pi M e^2}{m} \frac{\omega\gamma}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}$$

Define $\omega_p = \sqrt{\frac{4\pi M e^2}{m}}$ the "plasma frequency"




$$k = k_1 + ik_2 = \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\varepsilon_1 + i\varepsilon_2}$$

$$k^2 = k_1^2 - k_2^2 + 2ik_1k_2 = \frac{\omega^2}{c^2} \mu (\varepsilon_1 + i\varepsilon_2)$$

equate real and imaginary pieces and
solve for $k_1$ and $k_2$

$$k_1 = \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\sqrt{\varepsilon_1^2+\varepsilon_2^2} + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$k_2 = \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\sqrt{\varepsilon_1^2+\varepsilon_2^2} - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

**Regions of different behavior**

Regions ① and ④ – transparent propagation

$\varepsilon_1 > 0$ , $\varepsilon_1 \gg \varepsilon_2$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_1\left(1+\frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\varepsilon_1 + \frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_1}\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu\varepsilon_1} \quad + \text{ small correction}$$

$$k_2 \simeq \pm\frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_1\left(1+\frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm\frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_1}\right]^{1/2} = k_1\left(\frac{\varepsilon_2}{2\varepsilon_1}\right) \ll k_1$$

So $k_2 \ll k_1$ small attenuation

$\Rightarrow$ medium is transparent

Note: $v_p = \frac{\omega}{k_1} = \frac{c}{n} = \frac{c}{\sqrt{\varepsilon_1\mu}}$

in region ①, $\varepsilon_1 > 1 \Rightarrow v_p < c$

in region ④, $\varepsilon_1 < 1 \Rightarrow v_p > c$ !

but $v_g < c$ always!