Region (2) $\quad \omega \approx \omega_0 \quad$ <u>resonant absorption</u>

$$\varepsilon_2 \approx \frac{\omega_p^2}{\omega_0 \gamma} = \left(\frac{\omega_p}{\omega_0}\right)^2 \left(\frac{\omega_0}{\gamma}\right) \gg 1$$

for a sharp resonance with $\gamma \ll \omega_0$

$$\varepsilon_1 \approx 1$$

So $\quad \varepsilon_2 \gg \varepsilon_1$

$$k_1 \approx \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2\left(1 + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2\right) + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\approx \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2 + \frac{1}{4}\frac{\varepsilon_1^2}{\varepsilon_2} + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$k_1 \approx \pm \frac{\omega}{c}\sqrt{\mu}\sqrt{\frac{1}{2}\varepsilon_2}$$

$$k_2 \approx \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2\left(1 + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2\right) - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\approx \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2 + \frac{1}{4}\frac{\varepsilon_1^2}{\varepsilon_2} - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\approx \pm \frac{\omega}{c}\sqrt{\mu}\sqrt{\frac{1}{2}\varepsilon_2}$$

$k_1 \approx k_2 \quad$ <u>strong attenuation</u>

wave excites atoms at resonance $\Rightarrow$ large atomic displacements $\Rightarrow$ media absorbs most energy from the wave $\Rightarrow$ wave decays rapidly, decreases factor $\frac{1}{e^{2\pi}}$ within one wavelength of propagation.

Region 3 $\quad \epsilon_1 < 0, \quad |\epsilon_1| \gg \epsilon_2$

total reflection

width of region 3 is

$$\omega_1 - \omega_0 = \sqrt{\omega_0^2 + \omega_p^2} - \omega_0 \sim \omega_p \sim \sqrt{N}$$

increases with atomic density $\qquad$ as $\omega_p \gg \omega_0$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}|\epsilon_1| + \frac{1}{4}\frac{\epsilon_2^2}{|\epsilon_1|} + \frac{1}{2}\epsilon_1\right]^{1/2}$$

cancel as $|\epsilon_1| = -\epsilon_1$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu|\epsilon_1|}\;\frac{\epsilon_2}{2|\epsilon_1|}$$

$$k_2 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}|\epsilon_1| + \frac{1}{4}\frac{\epsilon_2^2}{|\epsilon_1|} - \frac{1}{2}\epsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu|\epsilon_1|}$$

$$\frac{k_2}{k_1} = \frac{2|\epsilon_1|}{\epsilon_2} \gg 1$$

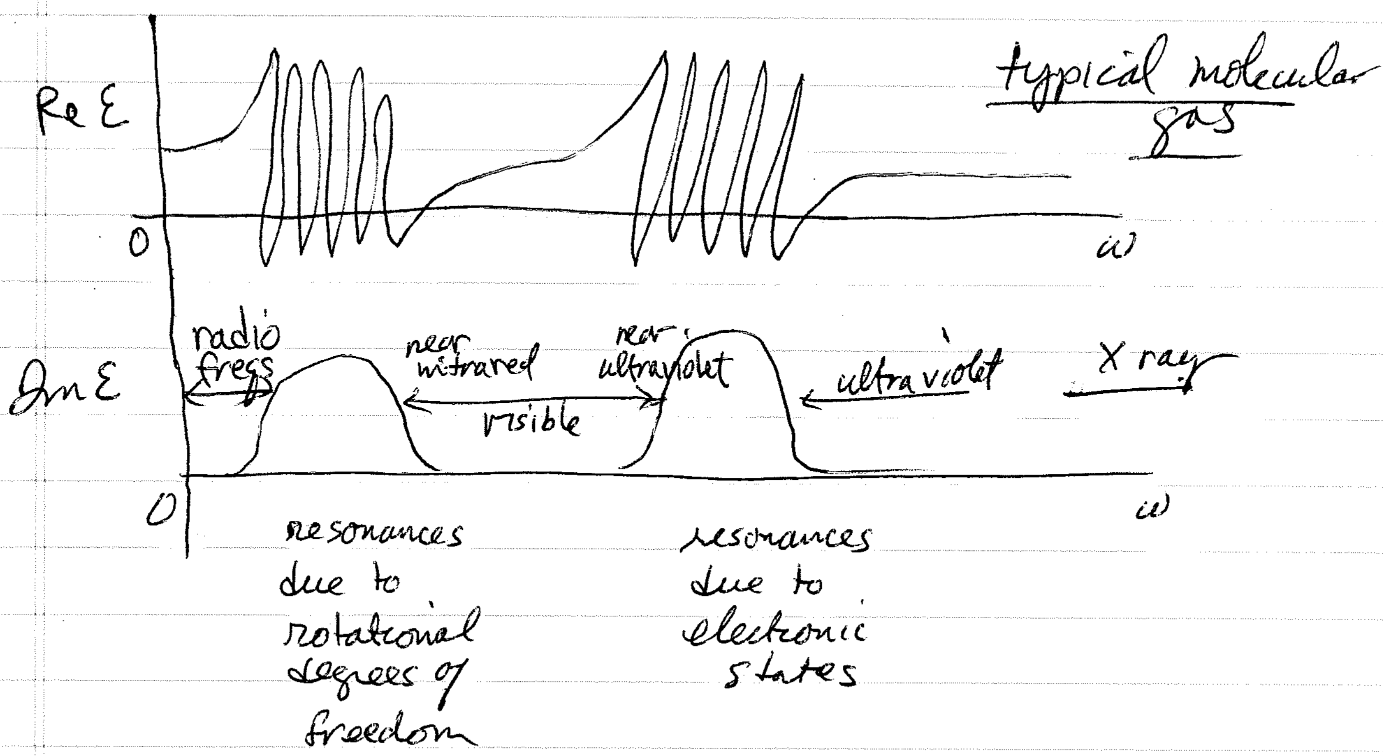
wave vector is almost pure imaginary
wave decays exponentially to zero in much less than one wavelength.

we will see this corresponds to total reflection
Since $\omega \gg \omega_0$, we are not at resonance so material is not absorbing much energy from wave. The strong attenuation is due to the destructive interference between the wave and the induced fields of the polarized atoms

Our simple model had a single resonance at ω_0
A more realistic model for molecules has many bands of resonances due to rotational, vibrational, and electronic modes of excitation

$$\varepsilon(\omega) = 1 + \omega_p^2 \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

where $\hbar\omega_i$ are spacings between energy levels with allowed electric dipole transitions



$$\omega_p = \sqrt{\frac{4\pi n e^2}{m}}$$

$$= 4.4 \times 10^{16} \sqrt{\frac{n}{n_A}}\ \sec^{-1}, \quad n_A = 6 \times 10^{23} / cm^3$$

~~For H_2O $\frac{n}{n_A}$~~

$$\Rightarrow \hbar\omega_p = 185 \sqrt{\frac{n}{n_A}}\ eV$$

For H_2O $\quad \frac{n}{n_A} \sim 0.05$

$$\hbar\omega_p \sim 40\ eV$$

For typical metal $\quad \frac{n}{n_A} \sim 0.1$

$$\hbar\omega_p \sim 58\ eV$$

compared to $\hbar\omega_0 \sim eV$

EM waves in Conductors

Conduction electrons are mobil, not bound
$\Rightarrow$ we have to include the $\vec{J}_f$ and ρ_f from them.

Simple Classical model for electron motion – "Drude Model"

$$m\ddot{\vec{r}} = -e\vec{E}(t) - \frac{m}{\tau}\dot{\vec{r}}$$

external E field

damping force due to collisions
τ is "relaxation time"

$$\vec{E} = \vec{E}_\omega e^{-i\omega t} \Rightarrow \vec{r} = \vec{r}_\omega e^{-i\omega t} \text{ solution}$$

plug in to get

$$\left(-\omega^2 - \frac{i\omega}{\tau}\right)\vec{r}_\omega = -\frac{e}{m}\vec{E}_\omega \Rightarrow \vec{r}_\omega = \frac{e}{m}\frac{1}{\omega^2 + \frac{i\omega}{\tau}}\vec{E}_\omega$$

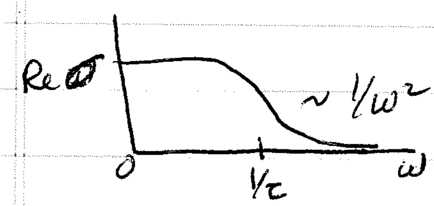
current is $\vec{J}_f = -en\dot{\vec{r}}_\omega = -en(-i\omega)\vec{r}_\omega$

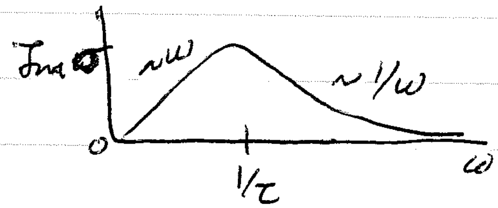
density of electrons

$$\vec{J}_f = \frac{ne^2}{m}\frac{i\omega}{\omega^2 + \frac{i\omega}{\tau}}\vec{E}_\omega = \frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau}\vec{E}_\omega$$

$$\vec{J}_f = \sigma(\omega)\,\vec{E}_\omega$$

conductivity $\boxed{\sigma(\omega) = \frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau}}$





$$\mathrm{Re}\,\sigma = \frac{\sigma_0}{1+\omega^2\tau^2} \qquad \mathrm{Im}\,\sigma = \frac{\sigma_0\,\omega\tau}{1+\omega^2\tau^2}$$

$$\sigma_0 = \sigma(0) = \frac{ne^2\tau}{m}$$ dc conductivity

Charge density ρ_f given by charge conservation law. For plane waves

$$\rho_f = \rho_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \quad \vec{J}_f = \vec{J}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla}\cdot\vec{J}_f \Rightarrow -i\omega\rho_\omega = -i\vec{k}\cdot\vec{J}_\omega$$

$$\rho_\omega = \frac{\vec{k}\cdot\vec{J}_\omega}{\omega} = \frac{\sigma(\omega)\,\vec{k}\cdot\vec{E}_\omega}{\omega}$$

Maxwell Equations

1) $\vec{\nabla}\cdot\vec{D} = 4\pi\rho_f$ 2) $\vec{\nabla}\cdot\vec{B} = 0$

3) $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ 4) $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J}_f + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$

Assume $\vec{H} = \vec{B}/\mu$, μ constant

$\vec{D}_\omega = \varepsilon_b(\omega)\vec{E}_\omega$ $\varepsilon_b(\omega)$ is dielectric function from the bound charges

$\vec{J}_\omega = \sigma(\omega)\vec{E}_\omega$ $\sigma(\omega)$ is conductivity from free charges

$\rho_\omega = \frac{\sigma}{\omega}\vec{k}\cdot\vec{E}_\omega$

<u>For harmonic plane wave solutions</u> $\vec{E} = \mathcal{E}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ etc.

1) $\Rightarrow \quad i\vec{k}\cdot\vec{D}_\omega = i\vec{k}\cdot\varepsilon_b \vec{E}_\omega = 4\pi \rho_\omega = 4\pi\sigma \dfrac{\vec{k}\cdot\vec{E}_\omega}{\omega}$

$\Rightarrow \quad i\vec{k}\cdot\vec{E}_\omega \left(\varepsilon_b + \dfrac{4\pi i\sigma}{\omega}\right) = 0$

2) $\Rightarrow \quad i\mu\,\vec{k}\cdot\vec{H}_\omega = 0$

3) $\Rightarrow \quad i\vec{k}\times\vec{E}_\omega = \dfrac{i\omega\vec{B}_\omega}{c} = \dfrac{i\omega\mu}{c} H_\omega$

4) $\Rightarrow \quad i\vec{k}\times\vec{H}_\omega = \dfrac{4\pi}{c}\vec{j}_\omega - \dfrac{i\omega}{c}\vec{D}_\omega$

$$= \frac{4\pi}{c}\sigma\vec{E}_\omega - \frac{i\omega}{c}\varepsilon_b\vec{E}_\omega$$

$$= -\frac{i\omega}{c}\left(\varepsilon_b + \frac{4\pi i\sigma}{\omega}\right)\vec{E}_\omega$$

Notice: all the equations above look <u>exactly</u> like what we had for the dielectric, provided we define

$$\boxed{\varepsilon(\omega) = \varepsilon_b(\omega) + \frac{4\pi i\sigma(\omega)}{\omega}}$$

So all results for the dielectric case carry over to conductors, provided we make the above substitution. In particular

dispersion relation for <u>transverse</u> modes $\boxed{k^2 = \dfrac{\omega^2}{c^2}\mu\,\varepsilon(\omega)}$

The main difference between dielectrics & conductors has to do with the contribution that the $4\pi i\sigma/\omega$ makes to the real and imaginary parts of $\varepsilon(\omega)$.

For simple Drude model $\sigma(\omega) = \dfrac{\sigma_0}{1-i\omega\tau}$ $\sigma_0 = \dfrac{ne^2\tau}{m}$

① <u>Low frequencies $\omega \ll 1/\tau$</u>

$\varepsilon_b(\omega) \approx \varepsilon_b(0)$ real

$\sigma(\omega) \approx \sigma_0$ real

$\Rightarrow \boxed{\varepsilon(\omega) \approx \varepsilon_b(0) + \dfrac{4\pi i\sigma_0}{\omega}}$ ← gives large ε_2 as $\omega \to 0$

$\Rightarrow$ strong dissipation

$\mathrm{Re}\,\varepsilon = \varepsilon_1$

$\mathrm{Im}\,\varepsilon = \varepsilon_2$

when $\dfrac{\varepsilon_2}{\varepsilon_1} = \dfrac{4\pi\sigma_0}{\omega\varepsilon_b(0)} \gg 1$ we call this regime a "<u>good</u>" conductor.
conduction electrons dominate the response
– waves strongly attenuated

when $\dfrac{\varepsilon_2}{\varepsilon_1} = \dfrac{4\pi\sigma_0}{\omega\,\varepsilon_b(0)} \ll 1$ we call this regime a "<u>poor</u>" conductor.
little absorption of energy by conduction electrons.
waves propagate

One always enters the "good" conductor region when ω gets sufficiently small.

wave vector:

$$k = \frac{\omega}{c}\sqrt{\mu\varepsilon}$$

for a good conductor where $\varepsilon_2 \gg \varepsilon_1$,

$$\varepsilon \sim i\varepsilon_2 = \frac{4\pi i \sigma_0}{\omega}$$

$$k = k_1 + i k_2 = \frac{\omega}{c}\sqrt{\mu \frac{4\pi i \sigma_0}{\omega}} \qquad \sqrt{i} = \frac{1+i}{\sqrt{2}}$$

$$k_1 = k_2 = \frac{\omega}{c}\sqrt{\frac{4\pi\mu\sigma_0}{2\omega}} = \frac{1}{c}\sqrt{2\pi\mu\sigma_0\omega}$$

for $\vec{k} = k\hat{z}$,

$$\vec{E} = \vec{E}_\omega e^{i(kz-\omega t)} = \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$$

$$\delta \equiv 1/k_2 = \frac{c}{\sqrt{2\pi\mu\sigma_0\omega}}$$

"skin depth" distance wave propagates into conductor

$$\delta \sim 1/\sqrt{\omega}$$

increases as ω decreases

ϕ phase shift between oscillations of $\vec{E}$ and $\vec{H}$

$$\phi = \arctan(k_2/k_1) \simeq \arctan(1) = 45^\circ$$

Amplitude ratio

$$\frac{|\vec{H}_\omega|}{|\vec{E}_\omega|} = \frac{c|k|}{\omega\mu} = \frac{\sqrt{2}\,c}{\omega\mu}k_1$$

$$= \frac{\sqrt{2}\,c}{\omega\mu}\frac{1}{c}\sqrt{2\pi\mu\sigma_0\omega}$$

$$= \sqrt{\frac{4\pi\sigma_0}{\omega\mu}} \sim 1/\sqrt{\omega}$$

as $\omega \to 0$, most of the energy of the wave is carried by the magnetic field part