(2) high frequencies $\omega > \frac{1}{\tau}, \omega > \omega_0$

$$\varepsilon_0(\omega) \approx 1$$

$$\sigma(\omega) \approx \frac{\sigma_0}{-i\omega} = \frac{i\omega e^2}{m\omega^2} = \frac{i\omega e^2}{m\omega^2}$$

pure imaginary

$$\varepsilon(\omega) \approx 1 + \frac{\sigma_0}{\omega} \leq 1 - \frac{4\pi e^2}{m\omega^2}$$

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

$$\omega_p = \sqrt{\frac{4\pi e^2}{m}}$$

plasma freq of the conduction electrons

$\varepsilon(\omega)$ is real

1) If $\omega > \omega_p$ then $\varepsilon > 0$

$\Rightarrow$ transparent propagation

$$k = k_1 = \frac{\omega}{c} \sqrt{\mu \varepsilon}$$

is pure real

$k_2 \equiv 0$

2) If $\omega < \omega_p$ then $\varepsilon < 0$

$\Rightarrow$ total reflection

$$k_1 \equiv 0$$

$$k = k_2 = \frac{\omega}{c} \sqrt{\mu \varepsilon(1)}$$

$k_2$ is pure imaginary

$\omega_p$ gives cross over between total reflection and transparent propagation
for typical metals
\[ T \approx 10^{-14} \text{ sec} \]
\[ \omega_p \approx 10^{16} \text{ sec}^{-1} \]
\[ \omega_p = \frac{2\pi C}{\lambda} \sim 3 \times 10^3 \text{ A} \] (visible \( \lambda \sim 5 \times 10^3 \text{ A} \))

**Example:** The ionosphere is a layer of charged gas surrounding the Earth. In many respects, the charged particles of the ionosphere behave like conduction electrons in a metal. The plasma freq of the ionosphere is such that

for AM radio \( \omega_m < \omega_p \Rightarrow \text{AM radio signals reflected back to Earth} \)

for FM radio \( \omega_M > \omega_p \Rightarrow \text{FM radio signals propagate through ionosphere into space} \)

Explains why you can pick up AM stations from far away - they get reflected back. But you can only pick up local FM stations.
Longitudinal modes in conductors

\[ \mathbf{E}_w \text{ or } \mathbf{H}_w \text{ not } \perp \mathbf{k} \]

magnetic field

\[ \nabla \cdot \mathbf{B} = 0 \Rightarrow \varepsilon \mu \mathbf{k} \cdot \mathbf{H}_w = 0 \Rightarrow \mathbf{H}_w \perp \mathbf{k} \text{ transverse} \]

\[ \text{or } \mathbf{E}_w = 0 \text{ spatially uniform } \mathbf{H} \]

\[ \text{If } \mathbf{E}_w = 0 \text{ then Faraday} \]

\[ \mathbf{i} \mathbf{k} \times \mathbf{E}_w = \varepsilon \omega \mathbf{H}_w = 0 \Rightarrow \mathbf{w} = 0 \]

\[ \text{as } \mathbf{k} = 0 \]

So only possible longitudinal \( \mathbf{H} \) is spatially uniform, constant in time.

electric field

\[ \nabla \cdot \mathbf{D} = \varepsilon \varepsilon_0 \rho_f \Rightarrow \varepsilon \varepsilon_0 (\mathbf{k} \cdot \mathbf{E}_w = 0 \Rightarrow \mathbf{E}_w \perp \mathbf{k} \text{ transverse} \]

\[ \mathbf{E}_w = 0 \]

If \( \mathbf{E}_w \parallel \mathbf{k} \) but \( \varepsilon \varepsilon_0 = 0 \), then can satisfy all other Maxwell equations.

\[ \mathbf{i} \mathbf{k} \times \mathbf{E}_w = \varepsilon \omega \mathbf{H}_w \Rightarrow \mathbf{H}_w = 0 \]

\[ \Rightarrow \mathbf{i} \omega \mathbf{k} \cdot \mathbf{H}_w = 0 \text{ and } \mathbf{i} \mathbf{k} \cdot \mathbf{H}_w = -\varepsilon \omega \varepsilon (\mathbf{k} \cdot \mathbf{E}_w) \]

\[ \text{as } \mathbf{H}_w = 0 \text{ as } \varepsilon (\mathbf{k} \cdot \mathbf{E}_w) = 0 \]

So we can have longitudinal electric field oscillation when \( \varepsilon (\mathbf{k} \cdot \mathbf{E}_w) = 0 \)
\( \text{low freq: } \omega \ll \omega_0, \quad \omega \ll \omega_1 \)

\[
E = E_0(0) + \frac{\text{i}4\pi e \sigma_0}{\omega}
\]

\[
E(\omega) = 0 \quad \text{when } \omega = -\frac{\text{i}4\pi e \sigma_0}{\varepsilon_0(0)}
\]

\[
\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{\text{i}(\mathbf{k} \cdot \mathbf{r} - \omega t)} = \mathbf{E}_0 e^{-\frac{\text{i}4\pi e \sigma_0}{\varepsilon_0(0)}} e^{-\text{i}k \cdot \mathbf{r}}
\]

If set up a longitudinal \( \bar{E} \) field, it decays to zero exponentially with a decay time \( \tau = \varepsilon_0(0)/4\pi \sigma_0 \). This is consistent with assumption the \( \bar{E} = 0 \) inside a conductor for electrostatics.

\( \bar{E} = \bar{E}_0 \Rightarrow \bar{E} \sim e^{-\text{i}k \cdot \mathbf{r}} \) in longitudinal.

\( \text{high freq: } \omega \gg \omega_c, \quad \omega \gg \omega_0 \)

\[
E(\omega) \approx 1 + \frac{4\pi e \sigma_0}{\omega} = 1 - \frac{e^2}{\omega_c^2} \quad \omega_c^2 = \frac{4\pi m e^2}{\varepsilon_0}
\]

\( E = 0 \) when \( \omega = \omega_c \)

So we have oscillatory longitudinal \( \bar{E} \) only when \( \omega = \omega_c \), independent of \( \mathbf{k} \).

\[
\mathbf{E} = \mathbf{E}_0 e^{\text{i}(\mathbf{k} \cdot \mathbf{r} - \omega_c t)}
\]

This is called a plasma oscillation. When one quantizes this oscillatory mode, it is called a plasmon.

\[
\nabla \cdot \mathbf{E} = 4\pi j \Rightarrow j = \frac{\text{i}k_0 E_0}{4\pi} e e^{-\text{i}k \cdot \mathbf{r} - \omega_c t}
\]

\( \mathbf{j} \) is a charge density oscillation.
Polarization

Consider a transverse plane wave traveling in direction \( \hat{m} \), i.e. \( \vec{k} = k \hat{m} \). Define a right-handed coordinate system as follows:

\[
\hat{e}_1 \times \hat{e}_2 = \hat{m} \\
\hat{m} \times \hat{e}_1 = \hat{e}_2 \\
\hat{e}_2 \times \hat{m} = \hat{e}_1
\]

A general solution to Maxwell's equations for a transverse plane wave is then

\[
\vec{E}(\vec{r}, t) = \text{Re} \left\{ (E_1 \hat{e}_1 + E_2 \hat{e}_2) e^{i(k \cdot \vec{r} - wt)} \right\}
\]

\[
\vec{H}(\vec{r}, t) = \frac{c}{\omega \mu} \text{Re} \left\{ k \hat{m} \times (E_1 \hat{e}_1 + E_2 \hat{e}_2) e^{i(k \cdot \vec{r} - wt)} \right\}
\]

\[
= \frac{c}{\omega \mu} \text{Re} \left\{ k (E_1 \hat{e}_2 - E_2 \hat{e}_1) e^{i(k \cdot \vec{r} - wt)} \right\}
\]

In general, \( k \) is complex:

\[
k = k_1 + ik_2 = |k| e^{i\delta}
\]

\[
\delta = \arctan \left( \frac{k_2}{k_1} \right)
\]

So far we implicitly assumed that \( E_1 \) and \( E_2 \) are real constants. In this case:

\[
\vec{E}(\vec{r}, t) = \vec{E}_0 e^{-k_2 \hat{m} \cdot \vec{r}} \cos (k_1 \hat{m} \cdot \vec{r} - wt)
\]

\[
\vec{H}(\vec{r}, t) = \vec{H}_0 e^{-k_2 \hat{m} \cdot \vec{r}} \cos (k_1 \hat{m} \cdot \vec{r} - wt + \delta)
\]

where

\[
\vec{E}_0 = E_1 \hat{e}_1 + E_2 \hat{e}_2 \quad \text{and} \quad \vec{H}_0 = \frac{c |k|}{\omega \mu} (E_1 \hat{e}_2 - E_2 \hat{e}_1)
\]

are fixed vectors for all time and space.
In this case the directions of \( \vec{E} \) and \( \vec{H} \) remain fixed while the amplitudes oscillate in time and space. Such a plane wave is called a linearly polarized wave.

However, there is nothing to prevent one from choosing a solution with \( E_1 \) and \( E_2 \) complex numbers:

\[
E_1 = |E_1| e^{iX_1} \quad E_2 = |E_2| e^{iX_2}
\]

In this case one has

\[
\vec{E}(\vec{r}, t) = \text{Re} \left\{ |E_1| \hat{e}_1 e^{i(\vec{k} \cdot \vec{r} - \omega t + X_1)} + |E_2| \hat{e}_2 e^{i(\vec{k} \cdot \vec{r} - \omega t + X_2)} \right\}
\]

\[
= e^{-i\frac{k^2 \hbar^2}{\omega}} \left[ |E_1| \hat{e}_1 \cos(k \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} - \omega t + X_1) + |E_2| \hat{e}_2 \cos(k \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} - \omega t + X_2) \right]
\]

and

\[
\vec{H}(\vec{r}, t) = \frac{c|k|}{\omega \mu} \text{Re} \left\{ |E_1| \hat{e}_2 e^{i(\vec{k} \cdot \vec{r} - \omega t + \delta + X_1)} - |E_2| \hat{e}_1 e^{i(\vec{k} \cdot \vec{r} - \omega t + \delta + X_2)} \right\}
\]

\[
= \frac{c|k|}{\omega \mu} e^{-i\frac{k^2 \hbar^2}{\omega}} \left[ |E_1| \hat{e}_2 \cos(k \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} - \omega t + \delta + X_1) - |E_2| \hat{e}_1 \cos(k \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} - \omega t + \delta + X_2) \right]
\]

Unless \( X_1 = X_2 \) we see that the components of \( \vec{E} \) and \( \vec{H} \) in directions \( e_1 \) and \( e_2 \) will oscillate out of phase with each other. Thus the directions of \( \vec{E} \) and \( \vec{H} \) will oscillate in time and space, as well as the amplitudes of \( \vec{E} \) and \( \vec{H} \). The direction of \( \vec{E} \) and \( \vec{H} \) is no longer fixed.
we will see that this situation in general corresponds to elliptically polarized wave!

**General case** $\vec{E}_1$ and $\vec{E}_2$ are complex constants

write $\vec{E}_1\hat{e}_1 + \vec{E}_2\hat{e}_2 = \vec{U} e^{i\phi}$

where $\phi$ is chosen so that $\vec{U} \cdot \vec{U}$ is real

- one can always do this since $\vec{U} \cdot \vec{U} = (E_1^2 + E_2^2) e^{-2i\phi}$

so $2\phi$ is just the phase of the complex $E_1^2 + E_2^2$

$\vec{U}$ is a complex vector $\Rightarrow \vec{U} = \vec{U}_a + i\vec{U}_b$

with $\vec{U}_a$ and $\vec{U}_b$ real vectors

Since $\vec{U} \cdot \vec{U}$ is real $\Rightarrow \vec{U}_a \cdot \vec{U}_b = 0$

so $\vec{U}_a \perp \vec{U}_b$ orthogonal

let $\hat{e}_a$ be the unit vector in direction of $\vec{U}_a$

so $\vec{U}_a = U_a \hat{e}_a$ with $U_a = |\vec{U}_a|$

let $\hat{e}_b = \hat{n} \times \hat{e}_a$ so that $\{\hat{m}, \hat{e}_a, \hat{e}_b\}$ are a right handed coordinate system

Then $\vec{U}_b = \pm U_b \hat{e}_b$ where $U_b = |\vec{U}_b|$

since $\vec{U}_b \perp \vec{U}_a$ and both are $\perp$ to $\hat{m}$.

It is $(\text{+})$ if $\vec{U}_b$ is parallel to $\hat{e}_b$ and

it is $(\text{-})$ if $\vec{U}_b$ is anti-parallel to $\hat{e}_b$. 
In this representation we have
\[ \tilde{E}(\mathbf{r}, t) = \text{Re} \left\{ U e^{i\gamma} e^{i(k\hat{r} \cdot \mathbf{r} - \omega t)} \right\} \]
\[ = e^{-k_z \hat{\mathbf{r}} \cdot \hat{z}} \text{Re} \left\{ U a \hat{e}_a e^{i(k\hat{\mathbf{\hat{r}}} \cdot \mathbf{r} - \omega t + \varphi)} \right\} \]
\[ \pm U b \hat{e}_b (e^{i(k\hat{\mathbf{\hat{r}}} \cdot \mathbf{r} - \omega t + \varphi)} \right\} \]
\[ = e^{-k_z \hat{\mathbf{r}} \cdot \hat{z}} \left\{ U a \hat{e}_a \cos (\varphi + \varphi) \pm U b \hat{e}_b \sin (\varphi + \varphi) \right\} \]

where we write \( \varphi = k_z \hat{r} \cdot \hat{z} - \omega t \)

Let's define \( e^{-k_z \hat{r} \cdot \hat{z}} U a \rightarrow U a \)
\( e^{-k_z \hat{r} \cdot \hat{z}} U b \rightarrow U b \)

so we don't have to keep writing the constant attenuation factor that is a common factor of all components of \( \tilde{E} \).

Then define \( E_a \) and \( E_b \) as the components of \( \tilde{E} \) in the directions \( \hat{e}_a \) and \( \hat{e}_b \) respectively.

\[ E_a = U a \cos (\varphi + \varphi) \]
\[ E_b = \mp U b \sin (\varphi + \varphi) \]

This then gives
\[ \left( \frac{E_a}{U a} \right)^2 + \left( \frac{E_b}{U b} \right)^2 = \cos^2 (\varphi + \varphi) + \sin^2 (\varphi + \varphi) = 1 \]

This is just the equation for an ellipse.
with semi-axes of lengths $U_a$ and $U_b$, oriented in the directions of $\hat{e}_a$ and $\hat{e}_b$.

$\Rightarrow$ At a fixed position $r$, the tip of the vector $\vec{E}$ will trace out the above ellipse as the time increases by one period of oscillation $\frac{2\pi}{\omega}$.

For (+), i.e. $\vec{U}_b = U_b \hat{e}_b$, $\vec{E}$ goes around the ellipse counterclockwise as $t$ increases

For (-), i.e. $\vec{U}_b = -U_b \hat{e}_b$, $\vec{E}$ goes around the ellipse clockwise as $t$ increases

Such a wave is said to be elliptically polarized

Special cases

1. $U_a = 0$ or $U_b = 0$
   the wave is linearly polarized
\( \hat{e}_a = \hat{e}_b \)

The tip of \( \vec{E} \) traces out a circle as \( t \) increases. The wave is circularly polarized.

The (+) case is said to have right-handed circular polarization.

The (-) case is said to have left-handed circular polarization.

One can define circular polarization basis vectors

\[
\hat{e}_+ = \frac{\hat{e}_a + i\hat{e}_b}{\sqrt{2}} \\
\hat{e}_- = \frac{\hat{e}_a - i\hat{e}_b}{\sqrt{2}}
\]

with \( \hat{e}_a \) and \( \hat{e}_b \) orthogonal.

A wave with amplitude \( \vec{E}_w = E \hat{e}_+ \) is right-handed circularly polarized.

A wave with complex amplitude \( \vec{E}_w = E \hat{e}_- \) is left-handed circularly polarized.

Just as the general case can always be written as a superposition of two orthogonal linearly polarized waves, i.e.

\[
\vec{E}_w = E_1 \hat{e}_1 + E_2 \hat{e}_2
\]
one can also always write the general case as a superposition of a left handed and a right handed circularly polarized wave

\[ \hat{U} = \hat{U}_a + i \hat{U}_b = Ua \hat{e}_a \pm iUb \hat{e}_b \]

\[ = (\frac{Ua + Ub}{\sqrt{2}}) \hat{e}_\pm + (\frac{Ua - Ub}{\sqrt{2}}) \hat{e}_\mp \]

(expand substitute in for \( \hat{e}_\pm \) and expand, to see that this is so)

\[ \Rightarrow \text{an elliptically polarized wave can be written as a superposition of circularly polarized waves} \]

As a special case of the above (if \( Ua = 0 \) or \( Ub = 0 \)) a linearly polarized wave can always be written as a superposition of circularly polarized waves.