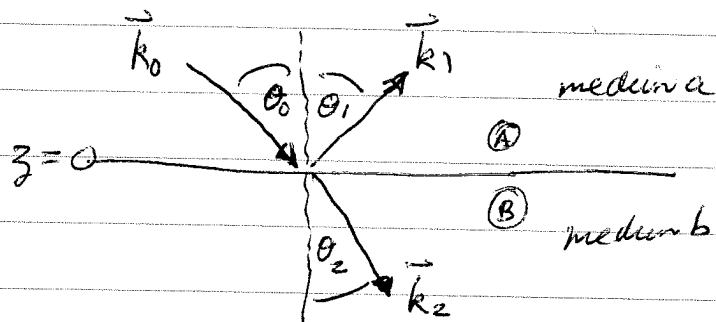


## Reflection & Transmission of waves at Interfaces



consider wave propagating from medium A into medium B.

for simplicity assume  $\epsilon_a$  is real and positive,  $\epsilon_b$  may be complex  
 $\mu_a$  and  $\mu_b$  are real and constant

$\vec{k}_0$  is incident wave,  $\theta_0 =$  angle of incidence

$\vec{k}_1$  is reflected wave,  $\theta_1 =$  angle of reflection

$\vec{k}_2$  is the transmitted or "refracted" wave,  $\theta_2 =$  angle of refraction

let each wave be given by

$$\vec{F}_n(\vec{r}, t) = \vec{F}_n e^{i(\vec{k}_n \cdot \vec{r} - \omega_n t)}$$

where  $\vec{F}_n$  can be either  $\vec{E}_n$  or  $\vec{H}_n$  for the electric or magnetic component of the wave

boundary condition: tangential component  $\vec{E}$  must be continuous at  $z=0$ . If  $\hat{x}$  is a vector in  $xy$  plane, and we consider  $\vec{r}=0$ , then

$$\Rightarrow \hat{x} \cdot \vec{E}_0 e^{-i\omega_0 t} + \hat{x} \cdot \vec{E}_1 e^{-i\omega_1 t} = \hat{x} \cdot \vec{E}_2 e^{-i\omega_2 t}$$

must be true for all time. can only happen if

$$\boxed{\omega_0 = \omega_1 = \omega_2 \equiv \omega} \quad \text{all frequencies are equal}$$

Now consider the same boundary condition for  $\vec{r}$  a position vector in the xy plane at  $z=0$ . Since  $\omega$ 's are all equal we can cancel out the common  $e^{-i\omega t}$  factors to get

$$\hat{x} \cdot \vec{E}_0 e^{i\vec{k}_0 \cdot \vec{r}} + \hat{x} \cdot \vec{E}_1 e^{i\vec{k}_1 \cdot \vec{r}} = \hat{x} \cdot \vec{E}_2 e^{i\vec{k}_2 \cdot \vec{r}}$$

this must be true for all  $\vec{r}$ . Can only happen if the projections of the  $\vec{k}_n$  in the xy plane are all equal

$$\boxed{\begin{aligned} k_{0x} &= k_{1x} = k_{2x} \\ k_{0y} &= k_{1y} = k_{2y} \end{aligned}}$$

only z components  $\vec{k}$  vectors can be different

Choose coord system as in diagram so that all  $\vec{k}$  vectors lie in the xz plane (y is out of page)

Since  $\epsilon_a$  is real and positive,  $\vec{k}_0$  and  $\vec{k}_1$  are real vectors

$$k_{0x} = k_{1x} \Rightarrow |\vec{k}_0| \sin \theta_0 = |\vec{k}_1| \sin \theta_1$$

$$\text{since } k_0^2 = \frac{\omega^2}{c^2} \mu \epsilon_a \quad \text{and } k_1^2 = \frac{\omega^2}{c^2} \mu \epsilon_a$$

$$\text{then } |\vec{k}_0| = |\vec{k}_1| \quad \text{so } \sin \theta_0 = \sin \theta_1$$

$$\boxed{\theta_0 = \theta_1}$$

angle of incidence = angle of reflection

If  $\epsilon_b$  is also real and positive (B is transparent)  
then  $|\vec{k}_2|$  is real

$$k_{0x} = k_{2x} \Rightarrow |\vec{k}_0| \sin \theta_0 = |\vec{k}_2| \sin \theta_2$$

$$k_2^2 = \frac{\omega^2}{c^2} \mu_b \epsilon_b$$

$$\Rightarrow \sqrt{\mu_a \epsilon_a} \sin \theta_0 = \sqrt{\mu_b \epsilon_b} \sin \theta_2$$

in terms of index of refraction  $n = \frac{kc}{\omega} = \frac{\omega \sqrt{\mu \epsilon}}{c}$

$$n = \sqrt{\mu \epsilon}$$

$$\Rightarrow n_a \sin \theta_0 = n_b \sin \theta_2$$

$$\boxed{\frac{\sin \theta_2}{\sin \theta_0} = \frac{n_a}{n_b}}$$

Snell's Law

true for all types of waves, not just EM waves

If  $n_a > n_b$  then  $\theta_2 > \theta_0$

In this case, when  $\theta_0$  is too large, we will have

$$\frac{n_a}{n_b} \sin \theta_0 > 1 \text{ as there will be no solution for } \theta_2$$

$\Rightarrow$  no transmitted wave

This is "total internal reflection" - wave does not exit medium A. The critical angle, above which one has total internal reflection, is given by

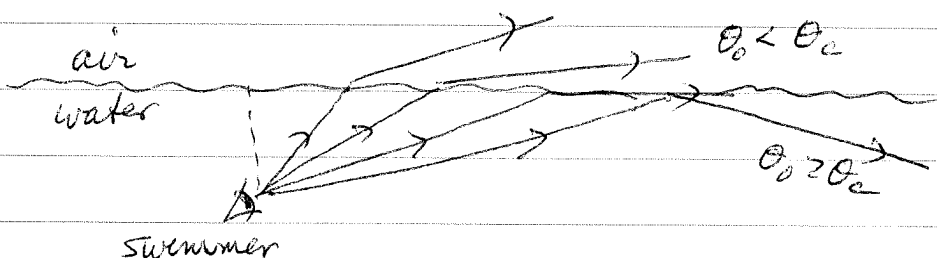
$$\frac{n_a}{n_b} \sin \theta_c = 1, \quad \boxed{\theta_c = \arcsin\left(\frac{n_b}{n_a}\right)}$$

$$\epsilon \sim 1 + 4\pi N \alpha \quad \leftarrow \text{density}$$

since  $n = \sqrt{\mu\epsilon}$  and  $\epsilon$  grows with density of the material, one usually has total internal reflection when one goes from a denser to a less dense medium.

Examples: diamonds sparkle due to total internal reflection. Diamonds have large  $n \Rightarrow$  small  $\theta_c \Rightarrow$  light bounces around inside many times before it can exit.

Can also see total internal reflection when swimming under water.



More general case  $\sqrt{\epsilon_b}$  is complex so  $\vec{k}_2$  is complex

$$\vec{k}_2 = \vec{k}_2' + i\vec{k}_2''$$

$$k_2' = |\vec{k}_2'|$$

$$k_2'' = |\vec{k}_2''|$$

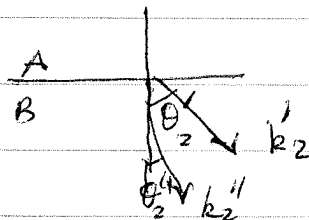
↑                    ↑  
real part    imaginary part

Note  $\vec{k}_2'$  and  $\vec{k}_2''$  need not be in the same direction!

condition  $k_{0x} = k_{2x} \Rightarrow \begin{cases} k_{0x} = k_{2x}' & \text{equate} \\ 0 = k_{2x}'' & \text{real and} \\ & \text{imaginary parts} \end{cases}$

$$k_0 \sin \theta_0 = k_2' \sin \theta_2'$$

$$0 = k_2'' \sin \theta_2''$$



$\Rightarrow \begin{cases} \theta_2'' = 0 \\ k_2'' = k_2'' \hat{z} \end{cases} \left\{ \begin{array}{l} \text{attenuation factor for the transmitted} \\ \text{wave is } e^{-k_2'' z} \end{array} \right.$

$\Rightarrow$  planes of constant amplitude are always parallel to the interface no matter what the angle of incidence  $\theta_0$ .

Having found  $\theta_2''$  there are still three quantities we must yet find in order to characterize the transmitted wave. These are  $\theta_2'$ ,  $k_2'$ ,  $k_2''$ .

To solve for these we will need 3 equations

one is:  $k_0 \sin \theta_0 = k_2' \sin \theta_2'$  ①

(from boundary condition)

where  $k_0 = \frac{\omega}{c} \sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c} n_a$  dispersion relation in medium a

The other two come from equating the real and imaginary parts of the dispersion relation in medium b.

$$k^2 = \frac{\omega^2}{c^2} \mu_b \epsilon_b = \frac{\omega^2}{c^2} \mu_b (\epsilon_{b1} + i \epsilon_{b2})$$

$$\begin{aligned}
 k^2 &= (\vec{k}_2' + i \vec{k}_2'') \cdot (\vec{k}_2' + i \vec{k}_2'') \\
 &= (k_2')^2 - (k_2'')^2 + 2i \vec{k}_2' \cdot \vec{k}_2''
 \end{aligned}$$

$$= (k_2')^2 - (k_2'')^2 + 2i k_2' k_2'' \cos \theta_2'$$

equating real and imaginary parts

$$(k_2')^2 - (k_2'')^2 = \frac{\omega^2}{c^2} \mu_b \epsilon_{b1} \quad (2)$$

$$2k_2' k_2'' \cos \theta_2' = \frac{\omega^2}{c^2} \mu_b \epsilon_{b2} \quad (3)$$

Use (2) and (3) to solve for  $k_2'$  and  $k_2''$  in terms of  $\theta_2'$

$$(2) \Rightarrow (k_2')^2 = (k_2'')^2 + \frac{\omega^2}{c^2} \mu_b \epsilon_{b1} \quad (4)$$

$$(3) \Rightarrow k_2'' = \frac{\omega^2}{c^2} \frac{\mu_b \epsilon_{b2}}{2k_2' \cos \theta_2'} \quad (5)$$

plug (5) into (4)

$$(k_2')^2 = \left( \frac{\omega^2}{c^2} \frac{\mu_b \epsilon_{b2}}{2k_2' \cos \theta_2'} \right)^2 + \frac{\omega^2}{c^2} \mu_b \epsilon_{b1}$$

$$\Rightarrow (k_2')^4 - \frac{\omega^2}{c^2} \mu_b \epsilon_{b1} (k_2')^2 - \frac{\omega^4}{c^4} \frac{\mu_b^2 \epsilon_{b2}^2}{4 \cos^2 \theta_2'} = 0$$

solve quadratic formula

$$(k_2')^2 = \frac{\omega^2 \mu_b \epsilon_{b1}}{2c^2} + \sqrt{\frac{\omega^4 \mu_b^2 \epsilon_{b1}^2}{4c^4} + \frac{\omega^4 \mu_b^2 \epsilon_{b2}^2}{4c^4 \cos^2 \theta_2'}}$$

take (+) solution only since  $(k_2')^2$  must be positive

$$= \frac{\omega^2 \mu_b}{c^2} \left[ \frac{\epsilon_{b1}}{2} + \frac{1}{2} \sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2 \theta_2'}} \right]$$

⑥

$$k_2' = \frac{\omega}{c} \sqrt{\mu_b} \left[ \frac{1}{2} \epsilon_{b1} + \frac{1}{2} \sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2 \theta_2'}} \right]^{1/2}$$

then get  $k_2''$  from ④

$$(k_2'')^2 = (k_2')^2 - \frac{\omega^2}{c^2} \mu_b \epsilon_{b1}$$

⑦

$$k_2'' = \frac{\omega}{c} \sqrt{\mu_b} \left[ -\frac{1}{2} \epsilon_{b1} + \frac{1}{2} \sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2 \theta_2'}} \right]^{1/2}$$

Note, these reduce to what we found earlier for the real and imaginary parts of the wave vector for a plane wave in a medium with complex  $\epsilon$ , IF we take  $\theta_2' = 0$ . We will have  $\theta_2'' = 0$  for normal incidence  $\theta_0 = 0$ .

Both  $k_2'$  and  $k_2''$  above still depend on the angle of refraction  $\theta_2'$ . We can close the set of equations by adding in Eq ①

$$k_0 \sin \theta_0 = k_2' \sin \theta_2'$$

⑧

$$\text{or } \frac{\omega}{c} m_a \sin \theta_0 = k_2' \sin \theta_2'$$

$$\text{where } m_a = \frac{k_0 c}{\omega} = \sqrt{\mu_a \epsilon_a}$$

Since the pair of equations ⑥ and ⑦ only involve the unknowns  $k_2'$  and  $\theta_2'$  we can

use them to eliminate  $k'_2$  and get a final single equation that determines  $\theta_2$

Define index of refraction in medium b

$$n_b = \sqrt{\mu_b \epsilon_b}$$

Then

$$\frac{\omega}{c} \mu_a \sin \theta_0 = \frac{\omega}{c} \mu_b \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{\epsilon_{b2}^2}{\epsilon_{b1}^2 \cos^2 \theta_2'}} \right]^{1/2} \sin \theta_2'$$

or

$$\mu_a \sin \theta_0 = \mu_b \sin \theta_2' \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{\epsilon_{b2}^2}{\epsilon_{b1}^2 \cos^2 \theta_2'}} \right]^{1/2}$$

This is the analog of Snell's law for propagation into a medium with complex dielectric function  $\epsilon$

### Cases

- ① For a nearly transparent material with  $\epsilon_{b2} \ll \epsilon_{b1}$  we can expand in  $\frac{\epsilon_{b2}}{\epsilon_{b1}}$  to get



$$m_a \sin \theta_0 = m_b \sin \theta_2' \left[ 1 + \frac{\epsilon_{b2}^2}{4 \epsilon_{b1}^2 \cos^2 \theta_2'} \right]^{1/2}$$

$$\approx m_b \sin \theta_2' \left[ 1 + \frac{\epsilon_{b2}^2}{8 \epsilon_{b1}^2 \cos^2 \theta_2'} \right]$$

↑  
small correction to  
Snell's law

for  $\frac{\epsilon_{b2}}{\epsilon_{b1}} \ll 1$  can solve iteratively

to lowest order:  $m_a \sin \theta_0 \approx m_b \sin \theta_2'$

$$\Rightarrow \cos^2 \theta_2' = 1 - \sin^2 \theta_2' = 1 - \left( \frac{m_a \sin \theta_0}{m_b} \right)^2$$

so to next order

$$m_a \sin \theta_0 \approx m_b \sin \theta_2' \left[ 1 + \frac{\epsilon_{b2}^2}{8 \epsilon_{b1}^2 \left( 1 - \frac{m_a^2}{m_b^2} \sin^2 \theta_0 \right)} \right]$$

$$\text{or } \sin \theta_2' \approx \frac{m_a \sin \theta_0}{m_b} \frac{1}{\left[ 1 + \frac{\epsilon_{b2}^2}{8 \epsilon_{b1}^2 \left( 1 - \frac{m_a^2}{m_b^2} \sin^2 \theta_0 \right)} \right]}$$

$$\leq \frac{m_a \sin \theta_0}{m_b}$$

result is that  $\theta_2'$  is smaller than Snell's law would predict.

② for a good conductor, or absorbing region of a dielectric,  $\epsilon_{b2} \gg \epsilon_{b1}$

to lowest order

$$n_a \sin \theta_0 = \sqrt{\mu_b \epsilon_{b1}} \left[ \frac{1}{2} \frac{\epsilon_{b2}}{\epsilon_{b1} \cos \theta_2'} \right]^{1/2} \sin \theta_2'$$

$$n_a \sin \theta_0 = \sqrt{\frac{\mu_b \epsilon_{b2}}{2}} \frac{\sin \theta_2'}{\sqrt{\cos \theta_2'}}$$

← very different from Snell's Law!

Snell's law only holds if both media are transparent