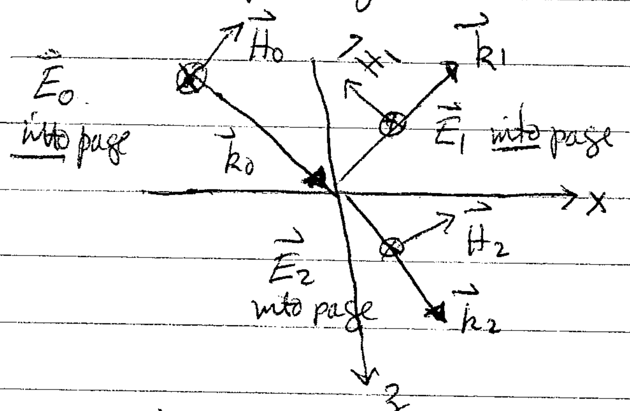
Reflection coefficients

Now we compute the amplitude of the reflected wave to determine how much of incident wave is reflected and how much is transmitted.

Consider two cases ① $\vec{E}_\omega$ is $\perp$ plane of incidence
② $\vec{E}_\omega$ lies in the plane of incidence

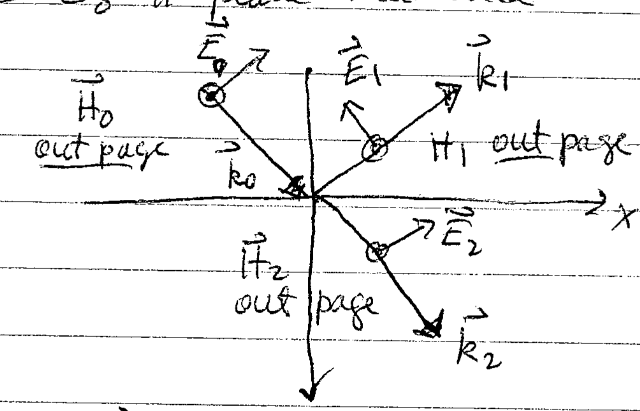
"plane of incidence" is the plane spanned by the wave vector $\vec{k}_0$ and the normal to the interface – in our case it is the xz plane

① $\vec{E}_0 \perp$ plane of incidence



$\Rightarrow \vec{H}_0$ in plane of incidence
all $\vec{E}$'s are in $\hat{y}$ direction

② $\vec{E}_0 \parallel$ plane incidence



$\Rightarrow \vec{H}_0 \perp$ plane of incidence
all the $\vec{H}$'s are in y direction

Continuity of y components

1) $E_0 + E_1 = E_2$

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continuity of x components

$$H_{0x} + H_{1x} = H_{2x}$$

Faraday

$$\frac{i\mu\omega}{c}\vec{H} = i\vec{k}\times\vec{E} \Rightarrow H_x = \frac{k_z c}{\omega\mu}E_y$$

$$E_{0x} + E_{1x} = E_{2x}$$

Ampere

$$-\frac{i\omega\epsilon}{c}\vec{E} = i\vec{k}\times\vec{H} \Rightarrow E_x = \frac{-k_z c}{\omega\epsilon}H_y$$

$\Rightarrow$

2) $\frac{k_{0z}}{\mu_a}(E_0 - E_1) = \frac{k_{2z}}{\mu_b} E_2$

solve (1) and (2) for E_1 and E_2 in terms of E_0

$$E_1 = \frac{\mu_b k_{0z} - \mu_a k_{2z}}{\mu_b k_{0z} + \mu_a k_{2z}} E_0$$

$$E_2 = \frac{2\mu_b k_{0z}}{\mu_a k_{2z} + \mu_b k_{0z}} E_0$$

2) $\frac{k_{0z}}{\epsilon_a}(H_0 - H_1) = \frac{k_{2z}}{\epsilon_b} H_2$

solve (1) and (2) for H_1 and H_2 in terms of H_0

$$H_1 = \frac{\epsilon_b k_{0z} - \epsilon_a k_{2z}}{\epsilon_b k_{0z} + \epsilon_a k_{2z}} H_0$$

$$H_2 = \frac{2\epsilon_b k_{0z}}{\epsilon_a k_{2z} + \epsilon_b k_{0z}} H_0$$

Define reflection coefficient in terms of the transported energy $R = \frac{|E_1|^2}{|E_0|^2} = \frac{|H_1|^2}{|H_0|^2}$

<u>Reflection coefficients</u>

① $\vec{E}_0 \perp$ plane incidence

$$R_\perp = \frac{|E_1|^2}{|E_0|^2} = \left| \frac{\mu_b k_{0z} - \mu_a k_{2z}}{\mu_b k_{0z} + \mu_a k_{2z}} \right|^2$$

② $\vec{E}_0 \parallel$ plane incidence

$$R_\parallel = \frac{|H_1|^2}{|H_0|^2} = \left| \frac{\epsilon_b k_{0z} - \epsilon_a k_{2z}}{\epsilon_b k_{0z} + \epsilon_a k_{2z}} \right|^2$$

Note: above are correct for an <u>arbitrary</u> medium B

i) Consider region of "total reflection"

$\Rightarrow$ $\left.\begin{array}{l} \text{Im}\,\epsilon_b = \epsilon_{b2} \approx 0 \\ \text{Re}\,\epsilon_b = \epsilon_{b1} < 0 \end{array}\right\} \Rightarrow \vec{k}_2 = i\vec{K}_2$ where $\vec{K}_2$ is real

ie $\vec{k}_2$ pure imaginary

$$\Rightarrow \quad R_\perp = \left| \frac{\mu_b k_{0z} - i\mu_a K_{2z}}{\mu_b k_{0z} + i\mu_a K_{2z}} \right|^2$$

$$R_\parallel = \left| \frac{\epsilon_b k_{0z} - i\epsilon_a K_{2z}}{\epsilon_b k_{0z} + i\epsilon_a K_{2z}} \right|^2$$

both are of the form $\left|\frac{a-ib}{a+ib}\right|^2 = 1$ when a, b real

$$\Rightarrow R_\perp = R_\parallel = 1$$

Confirms that the material is completely reflecting

ii) Next consider when medium B is transparent

ϵ_b is real and $\epsilon_b > 0$

$$k_{0z} = \frac{\omega}{c}\sqrt{\mu_a \epsilon_a}\cos\theta_0 = \frac{\omega}{c} n_a \cos\theta_0$$

$$k_{2z} = \frac{\omega}{c}\sqrt{\mu_b \epsilon_b}\cos\theta_2 = \frac{\omega}{c} n_b \cos\theta_2$$

Snell's law holds so $n_a \sin\theta_0 = n_b \sin\theta_2$

Can write $R_\perp$ and $R_\parallel$ as functions of θ_0

for simplicity take $\mu_a = \mu_b = 1$

(1) $$R_\perp = \left(\frac{n_a\cos\theta_0 - n_b\cos\theta_2}{n_a\cos\theta_0 + n_b\cos\theta_2}\right)^2 = \left(\frac{\cos\theta_0 - \left(\frac{\sin\theta_0}{\sin\theta_2}\right)\cos\theta_2}{\cos\theta_0 + \left(\frac{\sin\theta_0}{\sin\theta_2}\right)\cos\theta_2}\right)^2$$

$$= \left(\frac{\sin\theta_2\cos\theta_0 - \sin\theta_0\cos\theta_2}{\sin\theta_2\cos\theta_0 + \sin\theta_0\cos\theta_2}\right)^2$$

$$R_\perp = \left(\frac{\sin(\theta_0-\theta_2)}{\sin(\theta_0+\theta_2)}\right)^2$$

for $\theta_0 = 0$, ie normal incidence, $\theta_2 = 0$

$$\Rightarrow R_\perp = \left(\frac{n_a - n_b}{n_a + n_b}\right)^2$$ if $n_a = n_b$, no reflection! (not surprising!)

(2) $$R_{\parallel} = \left(\frac{\epsilon_b n_a\cos\theta_0 - \epsilon_a n_b\cos\theta_2}{\epsilon_b n_a\cos\theta_0 + \epsilon_a n_b\cos\theta_2}\right)^2$$ use $\sqrt{\epsilon_b} = n_b$, $\sqrt{\epsilon_a} = n_a$

$$= \left(\frac{n_b\cos\theta_0 - n_a\cos\theta_2}{n_b\cos\theta_0 + n_a\cos\theta_2}\right)^2$$

$$= \left(\frac{\cos\theta_0 - \left(\frac{\sin\theta_2}{\sin\theta_0}\right)\cos\theta_2}{\cos\theta_0 + \left(\frac{\sin\theta_2}{\sin\theta_0}\right)\cos\theta_2}\right)^2$$

$$= \left(\frac{\sin\theta_0\cos\theta_0 - \sin\theta_2\cos\theta_2}{\sin\theta_0\cos\theta_0 + \sin\theta_2\cos\theta_2}\right)^2$$

$$R_{\parallel} = \left(\frac{\tan(\theta_0-\theta_2)}{\tan(\theta_0+\theta_2)}\right)^2$$ ← after some algebra!

for $\theta_0 = 0$, then $\theta_2 = 0$

$$R_{\parallel} = \left(\frac{\epsilon_b n_a - \epsilon_a n_b}{\epsilon_b n_a + \epsilon_a n_b}\right)^2 = \left(\frac{n_b - n_a}{n_b + n_a}\right)^2 \quad \text{same as } R_{\perp}$$

So for $\theta_0 = 0$, $R_{\parallel} = R_{\perp}$ — this must be so since for $\theta_0 = 0$ there is no distinction between the $\perp$ and $\parallel$ cases

If $n_b = n_a$, $R_{\perp} = R_{\parallel} = 0$ no reflective wave

When $\theta_0 + \theta_2 = \pi/2$, then $\tan(\theta_0 + \theta_2) \to \infty$ and $R_{\parallel} = 0$

This occurs at an angle of incidence known as Brewster's angle θ_B, determined by

$$\underset{\substack{\uparrow \\ \theta_0}}{n_a \sin \theta_B} = n_b \sin\underset{\substack{\uparrow \\ \theta_2}}{\left(\frac{\pi}{2} - \theta_B\right)} = n_b \cos \theta_B$$

$$\Rightarrow \boxed{\tan \theta_B = \frac{n_b}{n_a}}$$

For incident wave at θ_B, reflected wave always has $\vec{E}_1 \perp$ plane of incidence, since $R_{\parallel} = 0$. If incoming wave has $\vec{E}_0 \parallel$ plane of incidence, then it gets completely transmitted. If $\vec{E}_0$ in general direction, reflected wave is always linearly polarized with $\vec{E}_1 \perp$ plane of incidence. — This is one method to create polarized light wave.

Kramers - Kronig Relation

We saw that $\vec{P}_\omega = \alpha(\omega)\,\vec{E}_\omega$

Causal response ie $\tilde{\alpha}(t) = 0$ for $t < 0$

$\Rightarrow$ $\alpha(\omega)$ has no poles in upper half of complex ω plane (UHP)

For any complex $\bar{\omega}$ in upper half of complex ω plane,

$$\alpha(\bar{\omega}) = \frac{1}{2\pi i}\oint \frac{\alpha(\omega')}{\omega'-\bar{\omega}}\,d\omega'$$ since no poles of α in UHP

contour along real axis, closed at infinity in UHP. The closing ~~top~~ semicircle at infinity gives no contribution assuming $\alpha(\omega)$ decays quickly enough as $|\omega| \to \infty$

$$\alpha(\bar{\omega}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \,\frac{\alpha(\omega')}{\omega'-\bar{\omega}}$$

Now consider $\bar{\omega} = \omega + i\delta$ where ω and δ are real and $\delta \to 0$

$$\alpha(\omega) = \lim_{\delta \to 0} \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \,\frac{\alpha(\omega')}{\omega'-\omega-i\delta}$$

Now $$\frac{1}{\omega'-\omega-i\delta} = P\left(\frac{1}{\omega'-\omega}\right) + i\pi\,\delta(\omega'-\omega)$$

principle part

$$\Rightarrow \alpha(\omega) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{\alpha(\omega')\, d\omega'}{\omega' - \omega}$$

$$\Rightarrow \boxed{\begin{aligned} \mathrm{Re}\, \alpha(\omega) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Im}\, \alpha(\omega')\, d\omega'}{\omega' - \omega} \\ \mathrm{Im}\, \alpha(\omega) &= \frac{-1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Re}\, \alpha(\omega')\, d\omega'}{\omega' - \omega} \end{aligned}}$$

Kramer Kronig relations

If know $\mathrm{Re}\,\alpha$ <u>or</u> $\mathrm{Im}\,\alpha$ can reconstruct full complex α

True for any causal response function