

## Radiation From moving Charges

In Lorentz gauge  $\frac{1}{c} \frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0$

$$\left. \begin{aligned} \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} &= -4\pi \rho \\ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} &= -\frac{4\pi}{c} \vec{j} \end{aligned} \right\} \begin{array}{l} \text{wave equation} \\ \text{with source} \end{array}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

If we can solve wave equation with source (inhomogeneous wave equation) then we are in principle done! To do this we want to find the Green's function for the wave equation

Recall from statics:  $\nabla^2 \phi = -4\pi \rho$

Green's function satisfies  $\nabla^2 G(\vec{r}) = -4\pi \delta(\vec{r})$

Then  $\phi(\vec{r}) = \int d^3r' G(\vec{r}-\vec{r}') \rho(\vec{r}') + \phi_0$   
 solution for infinite volume that vanishes as  $r \rightarrow \infty$  is  
 $G(\vec{r}-\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}$   $\nabla^2 \phi_0 = 0$

For wave equation we want solution to

$$\nabla^2 G(\vec{r}, t; \vec{r}', t') - \frac{1}{c^2} \frac{\partial^2 G(\vec{r}, t; \vec{r}', t')}{\partial t^2} = -4\pi \delta(\vec{r}-\vec{r}') \delta(t-t')$$

then we will have  $\left\{ \begin{array}{l} \phi(\vec{r}, t) = \int d^3r' G(\vec{r}, t; \vec{r}', t') \rho(\vec{r}', t') + \phi_0 \\ \vec{A}(\vec{r}, t) = \frac{1}{c} \int d^3r' G(\vec{r}, t; \vec{r}', t') \vec{j}(\vec{r}', t') + \vec{A}_0 \end{array} \right.$

where  $\nabla^2 \phi_0 - \frac{1}{c^2} \frac{\partial^2 \phi_0}{\partial t^2} = 0$  similarly for  $\vec{A}_0$

$\phi_0$  and  $\vec{A}_0$  could describe an incoming wave for example  
To construct the Green's function.

For infinite space (but not, for example, inside a cavity)

$$G(\vec{r}_t; \vec{r}'_t) = G(\vec{r} - \vec{r}', t - t')$$

express as Fourier transform

$$G(\vec{r}, t) = \int \frac{d^3k d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) G(\vec{r}, t) = \int \frac{d^3k d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\equiv \int \frac{d^3k d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) \left[-k^2 + \frac{\omega^2}{c^2}\right] e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$-4\pi \delta(\vec{r}) \delta(t) = -4\pi \int \frac{d^3k d\omega}{(2\pi)^4} e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

equating Fourier amplitudes

$$\Rightarrow \left[-k^2 + \frac{\omega^2}{c^2}\right] \tilde{G}(\vec{k}, \omega) = -4\pi$$

$$\tilde{G}(\vec{k}, \omega) = \frac{4\pi c^2}{k^2 c^2 - \omega^2}$$

when  $\omega^2 \neq c^2 k^2$

$$G(\vec{r}, t) = \int \frac{d^3k d\omega}{(2\pi)^4} \frac{4\pi c^2}{k^2 c^2 - \omega^2} e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

↑ poles at  $\omega = \pm ck$

In evaluating the  $\omega$  integral we have to know how

to treat the poles on the real axis so that  $G(\vec{r}, t)$  will have the desired behavior.

What we want is for  $G(\vec{r}, t)$  to be causal, i.e.  $G(\vec{r}, t) = 0$  for  $t < 0$ , so  $\phi(\vec{r}, t)$  and  $\vec{A}(\vec{r}, t)$  depend only on the values of the sources at earlier times  $t' < t$ .

$$\int d^3k e^{i\vec{k}\cdot\vec{r}} \tilde{G}(\vec{k}, \omega) = 2\pi \int_0^\pi d\theta \sin\theta \int_0^\infty dk k^2 e^{ikr \cos\theta} \tilde{G}(k, \omega)$$

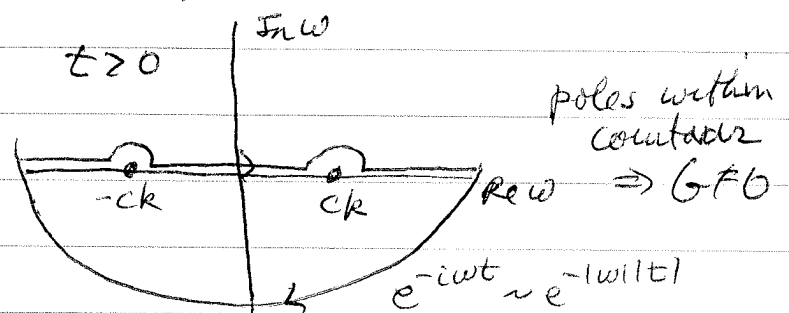
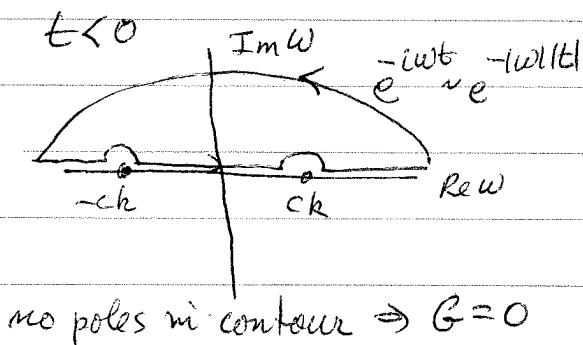
$$= 2\pi \int_{-1}^1 d\mu \int_0^\infty dk k^2 e^{ikr\mu} \tilde{G}(k, \omega) \quad \mu \equiv \cos\theta$$

$$= 4\pi \int_0^\infty dk k^2 \frac{\sin kr}{kr} \tilde{G}(k, \omega)$$

$$G(\vec{r}, t) = -\frac{c^2}{4\pi^2} \int_0^\infty dk k^2 \frac{\sin kr}{kr} \int_C \frac{e^{-i\omega t}}{(\omega + ck)(\omega - ck)} d\omega$$

↑ contour along real axis, but deformed to go around the poles

for  $t < 0$ ,  $e^{-i\omega t}$  will decay exponentially fast for large  $|\omega|$  in the upper half complex (UHP)  $\omega$  plane  $\Rightarrow$  can close contour in UHP for  $t < 0$ . If we want  $G = 0$  for  $t < 0$ , there should therefore be no poles in UHP. The contour  $C$  we want is therefore:



with this convention for the contour  $C$  we can evaluate the  $\omega$ -integral using Cauchy's residue theorem

$$\int \frac{e^{-i\omega t}}{(\omega+ck)(\omega-ck)} d\omega = -2\pi i \left[ \frac{e^{-ickt}}{2ck} - \frac{e^{ickt}}{2ck} \right]$$

$$= -\frac{2\pi \sin(ckt)}{ck}$$

$$G(\vec{r}, t) = \frac{2c}{\pi r} \int_0^\infty dk \sin(kr) \sin(ckt) = \frac{c}{\pi r} \int_{-\infty}^\infty dk \frac{(e^{ikr} - e^{-ikr})(e^{ickt} - e^{-ickt})}{(-4)}$$

$$= -\frac{c}{2r} \int_{-\infty}^\infty \frac{dk}{2\pi} \left[ e^{i(r+ct)k} + e^{-i(r+ct)k} - e^{i(r-ct)k} - e^{-i(r-ct)k} \right]$$

each integral would give a  $\delta$ -function, but for 1st two terms  $\delta(r+ct) = 0$  since here  $t \geq 0$  (by definition) and  $r = |\vec{r}| \geq 0$  so the argument will never vanish.

$$\boxed{G(\vec{r}, t) = \frac{c}{r} \delta(r-ct) = \frac{\delta(t - r/c)}{r}} \quad \text{using } \delta(ax) = \frac{1}{|a|} \delta(x)$$

$$G(\vec{r}, t, \vec{r}', t') = \begin{cases} \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} & t-t' \geq 0 \\ 0 & t-t' < 0 \end{cases} \quad \left. \begin{array}{l} \text{Green's function} \\ \text{for wave equation} \\ \text{in free space} \end{array} \right\}$$

$G \neq 0$  only on "light cone" that emanates from  $(r', t')$ , i.e. when  $|\vec{r}-\vec{r}'| = c(t-t')$ .  
Signal from source at  $(r', t')$  travels with  $c$ .

$$\phi(\vec{r}, t) = \phi_0(\vec{r}, t) + \int \frac{d^3r'}{t} \int_{-\infty}^t dt' \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} \rho(\vec{r}', t')$$

$$A(\vec{r}, t) = \vec{A}_0(\vec{r}, t) + \int \frac{d^3r'}{c} \int_{-\infty}^t dt' \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} \vec{j}(\vec{r}', t')$$

Apply to a single moving point charge

$$\rho(\vec{r}, t) = q \delta(\vec{r} - \vec{r}_0(t))$$

$$\vec{j}(\vec{r}, t) = q \vec{v}(t) \delta(\vec{r} - \vec{r}_0(t)) \quad \text{where } \vec{v}(t) = \frac{d\vec{r}_0}{dt}$$

Then

$$\phi(\vec{r}, t) = q \int dt' \frac{\delta(t-t' - \frac{1}{c} |\vec{r} - \vec{r}_0(t')|)}{|\vec{r} - \vec{r}_0(t')|}$$

because of the  $\vec{r}_0(t')$  in the argument of the  $\delta()$  function the  $t'$  dependence is not of the simple form  $t' - t_0$ .

We can write

$$g(t') \equiv t' + \frac{1}{c} |\vec{r} - \vec{r}_0(t')|$$

then

$$\phi(\vec{r}, t) = q \int dt' \frac{\delta(t - g(t'))}{|\vec{r} - \vec{r}_0(t')|}$$

$$= q \int \frac{\delta(t - g(t'))}{|\vec{r} - \vec{r}_0(t')|} dg \left( \frac{dg}{dt'} \right)$$

$$= \frac{q}{|\vec{r} - \vec{r}_0(t')|} \frac{1}{(dg/dt')} \Bigg|_{t' \text{ such that } g(t') = t}$$

$$g(t') = t' + \frac{1}{c} \sqrt{[x-x_0(t')]^2 + [y-y_0(t')]^2 + [z-z_0(t')]^2}$$

$$\frac{dg}{dt'} = 1 + \frac{1}{c |\vec{r}-\vec{r}_0(t')|} \left\{ [x-x_0(t')] \left( -\frac{dx_0}{dt'} \right) + \dots \right\}$$

$$= 1 - \frac{1}{c} \hat{n}(t') \cdot \vec{v}(t')$$

where  $\hat{n}(t') = \frac{\vec{r}-\vec{r}_0(t')}{|\vec{r}-\vec{r}_0(t')|}$

unit vector pointing from  $\vec{r}_0(t')$  to  $\vec{r}$

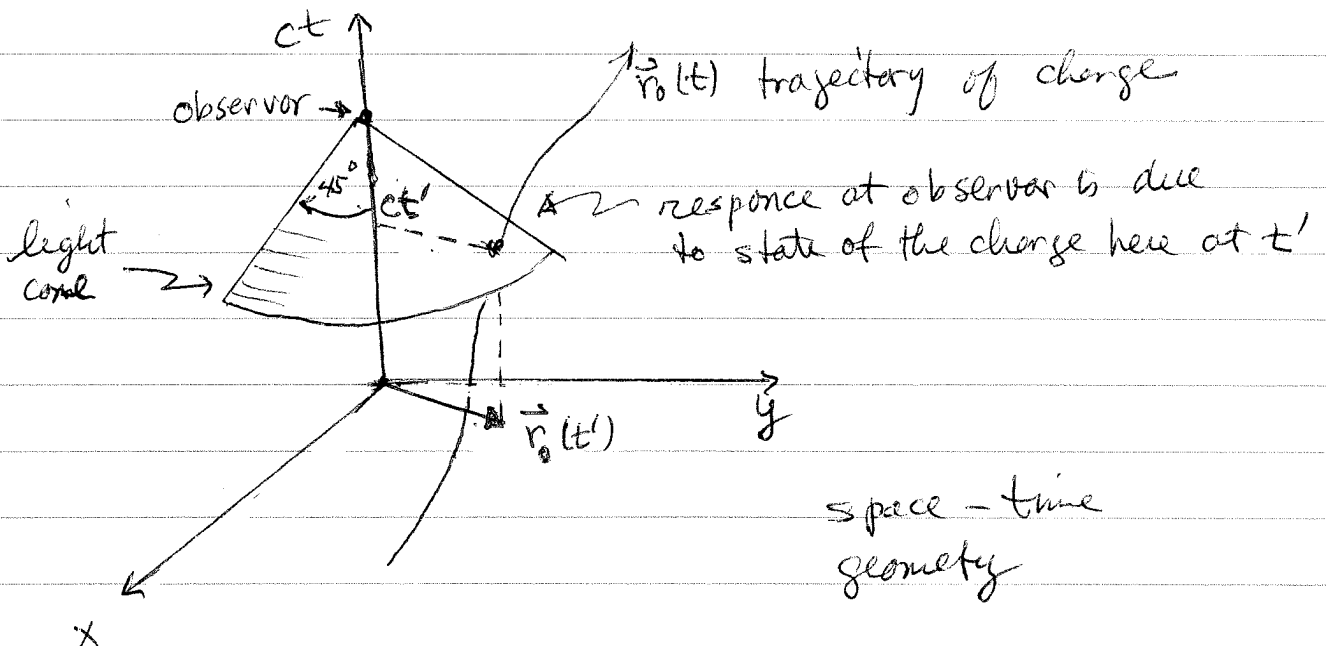
$$\phi(\vec{r}, t) = \frac{q}{|\vec{r}-\vec{r}_0(t')| \left[ 1 - \frac{1}{c} \hat{n}(t') \cdot \vec{v}(t') \right]}$$

$$\vec{A}(\vec{r}, t) = \frac{q \vec{v}(t')/c}{|\vec{r}-\vec{r}_0(t')| \left[ 1 - \frac{1}{c} \hat{n}(t') \cdot \vec{v}(t') \right]}$$

Liénard  
-Wiechert  
Potentials

where  $t'$  is determined by the condition

$$t - t' = \frac{1}{c} |\vec{r}-\vec{r}_0(t')|$$



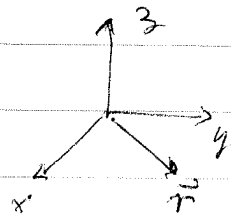
For charge moving with constant velocity along  $\hat{z}$

$$\vec{r}_0(t) = vt \hat{z} \quad \vec{v} = \frac{d\vec{r}_0}{dt} = v\hat{z}$$

For observer at position  $\vec{r}$  (in xy plane), time  $t$ , the fields will be determined by the charge at time  $t'$  such that

$$t - t' - \frac{|\vec{r} - \vec{r}_0(t')|}{c} = 0$$

$$t - t' - \frac{\sqrt{r^2 + v^2 t'^2}}{c} = 0$$



$$(t - t')^2 = t^2 + t'^2 - 2tt' = \frac{r^2 + v^2 t'^2}{c^2}$$

$$(1 - v^2/c^2) t'^2 - 2tt' + t^2 - r^2/c^2 = 0$$

$$\text{let } \gamma = (1 - v^2/c^2)^{-1/2}$$

$$t'^2 - 2\gamma^2 t t' + \gamma^2 (c^2 t^2 - r^2) = 0$$

$$t' = \gamma^2 t \pm \sqrt{\gamma^4 t^2 - \gamma^2 t^2 + \gamma^2 r^2/c^2}$$

$$= \gamma^2 t \pm \sqrt{\gamma^2 (\gamma^2 t^2 - t^2 + r^2/c^2)}$$

$$\gamma^2 - 1 = \frac{1}{1 - v^2/c^2} - 1 = \frac{v^2/c^2}{1 - v^2/c^2} = \gamma^2 \frac{v^2}{c^2}$$

$$= \gamma^2 t \pm \gamma \sqrt{t^2 \gamma^2 \left( \frac{v^2}{c^2} \right) + \frac{r^2}{c^2}}$$

$$t' = \gamma^2 t \pm \frac{\gamma}{c} \sqrt{v^2 t^2 + r^2}$$

consider  $t=0$ . solution should give  $t' < 0$   
 $\Rightarrow (-)$  sign is the solution we want

$$t' = \gamma^2 t - \frac{\gamma^2}{c} \sqrt{v^2 t^2 + r^2}$$

$$\phi(\vec{r}, t) = \frac{q}{|\vec{r} - \vec{r}_0(t')| \left[ 1 - \frac{1}{c} \dot{\vec{r}}(t') \cdot \vec{v} \right]}$$

~~$$|\vec{r} - \vec{r}_0(t')| = \sqrt{r^2 + v^2 t'^2}$$~~

$$|\vec{r} - \vec{r}_0(t')| = \sqrt{r^2 + v^2 t'^2} = c(t - t') \quad \leftarrow \text{from condition that determines } t'$$

$$(\vec{r} - \vec{r}_0(t')) \cdot \vec{v} = -\vec{r}_0(t') \cdot \vec{v} \quad \text{for } \vec{v} = v \hat{z}$$

$$= -v^2 t' \quad \vec{r} \text{ in } xy \text{ plane}$$

$$\phi(\vec{r}, t) = \frac{q}{c(t - t') \left[ 1 + \frac{v^2 t'}{c^2(t - t')} \right]}$$

$$= \frac{q}{c(t - t') + \frac{v^2 t'}{c}} = \frac{q}{c \left[ t - \left(1 - \frac{v^2}{c^2}\right) t' \right]}$$

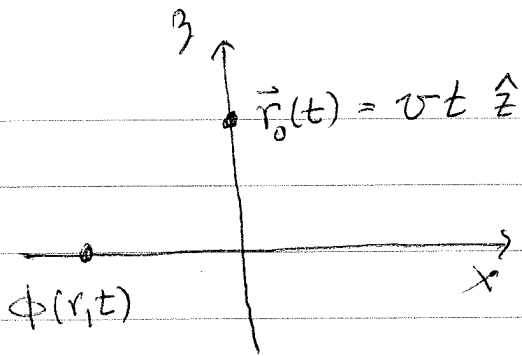
$$= \frac{q}{c \left( t - \frac{t'}{\gamma^2} \right)} = \frac{q}{c \frac{1}{\gamma^2} \sqrt{v^2 t^2 + r^2 / \gamma^2}}$$

$$\phi(\vec{r}, t) = \frac{q}{\sqrt{v^2 t^2 + r^2 / \gamma^2}}$$

$$\vec{A}(\vec{r}, t) = \frac{q \vec{v}}{c \sqrt{v^2 t^2 + r^2 / \gamma^2}}$$

solutions for  
 $\vec{r}$  in  $xy$  plane  
 when charge passes  
 through  $xy$  plane  
 at  $t=0$





at  $x$   
potential from charge at  $vt \hat{z}$

potential at pt  $\vec{r}$  in  $xy$  plane  
at time  $t$ , when charge is at  
 $\vec{r}_0 = vt \hat{z}$ , looks almost like  
static Coulomb potential, which  
would be  $\frac{q}{\sqrt{r^2 + v^2 t^2}}$

But instead, it is

$$\frac{q}{\sqrt{v^2 t^2 + \left(\frac{r}{\gamma}\right)^2}}$$

looks like the transverse direction has contracted  
by a factor  $\gamma$ !

Such considerations led Lorentz to discover  
the Lorentz transformation, before Einstein  
proposed his theory of special relativity