Radiation from a Localized Oscillating Source

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int d^3r' \, dt' \, \frac{\delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right)}{|\vec{r} - \vec{r}'|} \, \vec{J}(\vec{r}', t')$$

for pure harmonic oscillation in current

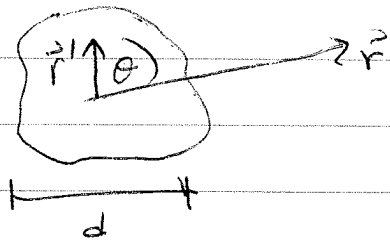$$\vec{J}(\vec{r}, t) = \text{Re}\left\{ \vec{J}_\omega(\vec{r}) \, e^{-i\omega t} \right\}$$

$$\Rightarrow \vec{A}(\vec{r}, t) = \text{Re}\left\{ \vec{A}_\omega(\vec{r}) \, e^{-i\omega t} \right\}$$

$$\Rightarrow \vec{A}_\omega(\vec{r}) e^{-i\omega t} = \frac{1}{c} \int d^3r' \, \vec{J}_\omega(\vec{r}') \, e^{-i\omega t} \, \frac{e^{i\omega |\vec{r} - \vec{r}'|/c}}{|\vec{r} - \vec{r}'|}$$

doing $\int dt'$ by using the δ-function

$$\vec{A}_\omega(\vec{r}) = \frac{1}{c} \int d^3r' \, \vec{J}_\omega(\vec{r}') \, \frac{e^{i\omega |\vec{r} - \vec{r}'|/c}}{|\vec{r} - \vec{r}'|}$$

Assume source is localized, ie $\vec{J}_\omega(\vec{r}) \approx 0$ for $|\vec{r}| > d$

Approx ①

for $r \gg d$, _far from sources_

$$|\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2rr' \cos\theta}$$

$$= r \sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r} \cos\theta}$$

$$\approx r\left(1 - \frac{r'}{r} \cos\theta\right)$$

$$\approx r - \vec{r}' \cdot \hat{r} + o\left(\frac{r'}{r}\right)^2$$

$$\hat{r} \equiv \frac{\vec{r}}{r}$$

$$\vec{A}_\omega(\vec{r}) = \frac{1}{c} \int d^3r' \; \frac{\vec{f}_\omega(\vec{r}')}{r - \vec{r}' \cdot \hat{r}} \; e^{ik(r - \vec{r}' \cdot \hat{r})} \qquad \text{where } k \equiv \frac{\omega}{c}$$

$$= \frac{e^{ikr}}{cr} \int d^3r' \; \frac{\vec{f}_\omega(\vec{r}') \, e^{-ik\vec{r}' \cdot \hat{r}}}{1 - \frac{\hat{r} \cdot \vec{r}'}{r}}$$

$$\approx \frac{e^{ikr}}{cr} \int d^3r' \; \vec{f}_\omega(\vec{r}') \, e^{-ik\hat{r} \cdot \vec{r}'} \left(1 + \frac{\hat{r} \cdot \vec{r}'}{r} \right)$$

when combine with the $e^{-i\omega t}$ piece, this gives outgoing spherical wave $\dfrac{e^{i(kr - \omega t)}}{r}$

oscillating charge radiates outgoing spherical electromagnetic waves

the $\int d^3r' \; \vec{f}_\omega(\vec{r}') \cdots$ term will determine the angular dependence of the radiation.

Approx ② $\lambda \gg d$ long wavelength approx

or $kd \ll 1$ $\Rightarrow \frac{\omega}{c} d \ll 1$ or $\frac{d}{\tau} \ll c$

where τ is period of oscillation.

Since $\frac{d}{\tau}$ is max speed of the oscillating charges $\Rightarrow \lambda \gg d$ is a non-relativistic approximation

$$kd \ll 1 \implies e^{-ik\hat{r}\cdot\vec{r}'} \approx 1 - ik\hat{r}\cdot\vec{r}' + \text{higher orders}$$

$$\vec{A}_\omega(\vec{r}) = \frac{e^{ikr}}{cr} \int d^3r' \, \vec{j}_\omega(\vec{r}') \left(1 - ik\hat{r}\cdot\vec{r}'\right)\left(1 + \frac{\hat{r}\cdot\vec{r}'}{r}\right)$$

$$= \frac{e^{ikr}}{cr} \int d^3r' \, \vec{j}_\omega(\vec{r}') \left[1 + \hat{r}\cdot\vec{r}'\left(\frac{1}{r} - ik\right)\right]$$

$$+ \text{higher order in } \frac{d}{r} \text{ or } kd$$

$$\vec{A}_\omega(\vec{r}) = \frac{e^{ikr}}{r}\left[\ \vec{I}_1 + \left(\frac{1}{r} - ik\right)\vec{I}_2\ \right]$$

where $\quad \vec{I}_1 \equiv \frac{1}{c}\int d^3r' \, \vec{j}_\omega(\vec{r}')$

$$\vec{I}_2 \equiv \frac{1}{c}\int d^3r' \, \hat{r}\cdot\vec{r}' \, \vec{j}_\omega(\vec{r}')$$

<u>Consider first $\vec{I}_1$</u> $\quad i$th component $\qquad (\vec{I}_1$ vanishes in statics$)$

$$\int d^3r \, j_i(\vec{r}) = -\int d^3r \, r_i \, \vec{\nabla}\cdot\vec{j} \qquad \text{integration by parts}$$

$$= \int d^3r \, r_i \, \frac{\partial \rho}{\partial t} \qquad \text{as } \vec{\nabla}\cdot\vec{j} + \frac{\partial \rho}{\partial t} = 0$$

$$\int d^3r \, j_i(\vec{r}) = -i\omega \int d^3r \, r_i \, \rho_\omega(\vec{r})$$

$$\implies \vec{I}_1 = -\frac{i\omega}{c}\int d^3r \, \vec{r}\,\rho_\omega(\vec{r}) = -\frac{i\omega\vec{P}_\omega}{c}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underset{\text{dipole moment}}{\overset{\text{electric}}{\uparrow}}$

Electric dipole approximation from $\vec{I}_1$

$$\boxed{\vec{A}_{E1}(\vec{r}) = \frac{e^{ikr}}{r}\left(-\frac{i\omega}{c}\vec{p}_\omega\right) = -i\,\vec{p}_\omega\,k\,\frac{e^{ikr}}{r}}$$

$\omega = ck$

Consider $\vec{I}_2$

$$\vec{I}_2 = \frac{1}{c}\int d^3r'\;\hat{r}\cdot\vec{r}'\;\vec{j}_\omega(\vec{r}') = \frac{1}{c}\hat{r}\cdot\int d^3r'\;\vec{r}'\,\underbrace{\vec{j}_\omega(\vec{r}')}_{\text{tensor}}$$

we saw this tensor earlier when we did the
magnetic dipole approx, and when we derived the
macroscopic Maxwell equations

$$\int d^3\vec{r}'\;\vec{r}'\vec{j}_\omega(\vec{r}') = -\int d^3r'\;\vec{j}_\omega(\vec{r}')\vec{r}' - \int d^3r'\;\vec{r}'\vec{r}'\left(\vec{\nabla}'\cdot\vec{j}_\omega(r')\right)$$

$$= \frac{1}{2}\int d^3r'\left[\vec{r}'\vec{j}_\omega - \vec{j}_\omega\vec{r}'\right] - \frac{1}{2}\int d^3r'\;i\omega\,\vec{r}'\vec{r}'\rho_\omega$$

using $\vec{\nabla}'\cdot\vec{j} = -\frac{\partial\rho}{\partial t}$

$$\vec{I}_2 = \frac{1}{2c}\int d^3r'\left[(\hat{r}\cdot\vec{r}')\vec{j}_\omega - (\hat{r}\cdot\vec{j}_\omega)\vec{r}'\right] - \frac{1}{2}\frac{i\omega}{c}\hat{r}\cdot\int d^3r'\,(\vec{r}'\vec{r}')\rho_\omega(\vec{r}')$$

$$= -\frac{1}{2c}\int d^3r'\left[\hat{r}\times(\vec{r}'\times\vec{j}_\omega)\right] - \frac{1}{2}\frac{i\omega}{c}\hat{r}\cdot\int d^3r'\,(\vec{r}'\vec{r}')\rho_\omega(\vec{r}')$$

$$= -\hat{r}\times\vec{m}_\omega - \frac{1}{2}\frac{i\omega}{3c}\hat{r}\cdot\overleftrightarrow{Q}'_\omega$$

where $\vec{m}_\omega = \frac{1}{2c}\int d^3r'\;\vec{r}'\times\vec{j}_\omega(\vec{r}')$ is magnetic dipole moment

$\overleftrightarrow{Q}'_\omega = \int d^3r'\;3\vec{r}'\vec{r}'\,\rho_\omega(\vec{r}')$ looks almost like electric
quadrapole tensor

to make it look like the proper quadrapole moment

$$\overleftrightarrow{Q}_\omega = \int d^3r'\left(3\vec{r}\,'\vec{r}\,' - r'^2\overleftrightarrow{I}\right)\rho_\omega(\vec{r}\,')$$

we can write

$$\overleftrightarrow{Q}'_\omega = \overleftrightarrow{Q}_\omega + \overleftrightarrow{I}\int d^3r'\ r'^2\rho_\omega(\vec{r}\,')$$

$\qquad\qquad\qquad\underset{\text{identity matrix } I_{ij} = \delta_{ij}}{\uparrow}$

$$\vec{I}_2 = -\hat{r}\times\vec{m}_\omega - \frac{1}{2}\frac{i\omega}{3c}\hat{r}\cdot\overleftrightarrow{Q}_\omega - \frac{i\omega}{6c}\hat{r}\,C(\omega)$$

$\qquad\qquad\qquad\qquad\qquad\qquad$ where $C_\omega \equiv \int d^3r'\,r'^2\rho_\omega(\vec{r}\,')$

$\qquad\qquad\qquad\qquad\qquad\qquad$ is a scalar

<u>Magnetic dipole approximation from $\vec{I}_2$</u>

$$\boxed{\vec{A}_{M1}(\hat{r}) = \frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\left(-\hat{r}\times\vec{m}_\omega\right)}$$

<u>Electric quadrapole approximation from $\vec{I}_2$</u>

$$\boxed{\vec{A}_{E2}(\hat{r}) = \frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\left(-\frac{i\omega}{6c}\hat{r}\cdot\overleftrightarrow{Q}_\omega\right)}$$

The last piece $\dfrac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\left(-\frac{i\omega}{6c}\hat{r}\,C_\omega\right)$

$\qquad$ can always be ignored — it is a radial function
and so its curl always vanishes $\Rightarrow$ gives
no contribution to $\vec{B}$. Similarly, since $-\frac{i\omega}{c}\vec{E}_\omega = i\vec{k}\times\vec{B}_\omega$
by Ampere's law, this term will give no $\qquad\uparrow$
contribution to $\vec{E}$. $\qquad\qquad\qquad\qquad\qquad$ holds away
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ from source
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ where $\vec{J}\approx 0$.

So   with these two approximations ① ad ②

$$\vec{A}_\omega(\vec{r}) = \vec{A}_{E1}(\vec{r}) + \vec{A}_{M1}(\vec{r}) + \vec{A}_{E2}(\vec{r})$$

keeping higher order terms would give magnetic quadrapole, electric octopole etc.

Compare strengths of the terms above

Approx ③   Radiation zone: far from sources,
$(r \gg \lambda)$ $\frac{1}{r} \ll k$ so $(\frac{1}{r} - ik) \approx -ik$ in $\vec{A}_{M1}$ and $\vec{A}_{E2}$
only keep terms of $O(\frac{1}{r})$

electric dipole $\vec{P}_\omega \sim qd$ $\vec{A}_{E1} \sim qkd$

magnetic dipole $\vec{m}_\omega \sim \frac{v q d}{c}$ $\vec{A}_{M1} \sim qkd(\frac{v}{c})$

use $v \sim \frac{d}{t} \sim d\omega \sim dkc \Rightarrow \vec{A}_{M1} \sim q(kd)^2$

electric quadrapole $\overset{\leftrightarrow}{Q}_\omega \sim qd^2$ $\vec{A}_{E2} \sim qd^2 k \frac{\omega}{c} \sim q(kd)^2$

Since   Approx ②   assumed $kd \approx \frac{v}{c} \ll 1$
above is expansion in powers of kd

leading term is electric dipole
next order are $\left[\begin{array}{l} \text{magnetic dipole} \\ \text{electric quadrapole} \end{array}\right]$ $\frac{A_{M1}}{A_{E1}} \sim \frac{A_{E2}}{A_{E1}} \sim kd$

next order terms are smaller than A_{E1} by factor $(kd)^2$
etc.

Electric dipole Approximation — the leading term in non-relativistic expansion

$$\vec{A}_{E1}(\vec{r}) = -ik\,\vec{p}_\omega\,\frac{e^{ikr}}{r} \qquad\qquad \vec{\nabla}\times(\varphi\vec{F}) = (\vec{\nabla}\varphi)\times\vec{F} + \varphi\vec{\nabla}\times\vec{F}$$

$$\vec{B}_{E1} = \vec{\nabla}\times\vec{A}_{E1} = -ik\left(\vec{\nabla}\,\frac{e^{ikr}}{r}\right)\times\vec{p}_\omega$$

$$= -ik\left(ik - \frac{1}{r}\right)\frac{e^{ikr}}{r}\,\hat{r}\times\vec{p}_\omega$$

$$= k^2\frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)\hat{r}\times\vec{p}_\omega$$

In radiation zone approx, $kr \gg 1$

$$\boxed{\vec{B}_{E1} \simeq k^2\frac{e^{ikr}}{r}\,\hat{r}\times\vec{p}_\omega}$$

To get electric field, use Ampere's Law

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \qquad \text{(away from source where } \vec{J} = 0\text{)}$$

For oscillating fields $\vec{E} = E_\omega e^{-i\omega t}$

$$\vec{\nabla}\times\vec{B}_\omega = -\frac{i\omega}{c}\vec{E}_\omega \qquad\Rightarrow\qquad E_{E1} = \frac{i}{k}\vec{\nabla}\times\vec{B}_{E1}$$

$$\vec{E}_{E1} = \frac{i}{k}\vec{\nabla}\times\left[k^2\frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)\hat{r}\times\vec{p}_\omega\right]$$

$$\vec{E}_{E1} = \frac{i}{k} \left(\vec{\nabla} e^{ikr} \right) \times \left[\frac{k^2}{r} \left(1 + \frac{i}{kr} \right) \hat{r} \times \vec{p}_\omega \right]$$

$$+ \frac{i}{k} e^{ikr} \vec{\nabla} \times \left[\frac{k^2}{r} \left(1 + \frac{i}{kr} \right) \hat{r} \times \vec{p}_\omega \right]$$

this will always be of order $1/r^2$

so can ignore it in radiation zone approx

So in <u>radiation zone approx</u>

$$\vec{E}_{E1} = \left(\vec{\nabla} e^{ikr} \right) \times \left[\frac{ik}{r} \hat{r} \times \vec{p}_\omega \right]$$

$$\boxed{\vec{E}_{E1} = -k^2 \frac{e^{ikr}}{r} \hat{r} \times \left(\hat{r} \times \vec{p}_\omega \right)}$$
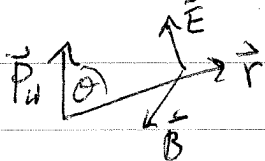
if do not make radiation zone approx, one gets

$$\vec{E}_{EI} = \frac{k^2 e^{ikr}}{r} \left[\vec{p}_\omega - \hat{r}(\vec{p}_\omega \cdot \hat{r}) - \frac{i}{kr}\left(1 + \frac{i}{kr}\right)\left(3\hat{r}(\vec{p}_\omega \cdot \hat{r}) - \vec{p}\right)\right]$$

Using radiation zone approx:

$$\vec{E}_{EI} = k^2 \frac{e^{ikr}}{r} \hat{r} \times (\vec{p}_\omega \times \hat{r}) \qquad\qquad |\vec{E}_{EI}| = |\vec{B}_{EI}|$$

$$\vec{B}_{EI} = -k^2 \frac{e^{ikr}}{r} \vec{p}_\omega \times \hat{r} \qquad\qquad \vec{E}_{EI} \perp \vec{B}_{EI}$$

If choose coordinates so that $\vec{p}_\omega$ is along $\hat{z}$ axis, then

$$\vec{E}_{EI} = -k^2 p_\omega \frac{e^{ikr}}{r} \sin\theta \, \hat{\theta}$$

$$\vec{B}_{EI} = -k^2 p_\omega \frac{e^{ikr}}{r} \sin\theta \, \hat{\varphi}$$

<u>Emitted power</u>

Poynting vector $\quad \vec{S}_{EI}(\vec{r},t) = \frac{c}{4\pi} Re\{\vec{E}_{EI}\} \times Re\{\vec{B}_{EI}\}$

need to take real parts of complex expression before multiplying

$$Re\{\vec{E}_{EI}(\vec{r},t)\} = -k^2 p_\omega \frac{\cos(kr-\omega t)}{r} \sin\theta \, \hat{\theta}$$

$$Re\{\vec{B}_{EI}(\vec{r},t)\} = -k^2 p_\omega \frac{\cos(kr-\omega t)}{r} \sin\theta \, \hat{\varphi}$$

$$\boxed{\vec{S}_{EI}(\vec{r},t) = \frac{c}{4\pi} k^4 p_\omega^2 \frac{\cos^2(kr-\omega t)}{r^2} \sin^2\theta \, \hat{r}}$$