$\vec{S}_{E1} \sim \hat{r} \Rightarrow$ energy flows radially outwards

$\vec{S}_{E1} \sim \frac{1}{r^2} \Rightarrow$ energy conserved

$$\oint_{\substack{\text{sphere}\\ \text{radius } R}} da\, \hat{n} \cdot \langle \vec{S}_{E1} \rangle = \text{constant for all } R$$

— Question - what about the $\frac{1}{r^n}$, $n \geq 2$, terms if we do not make radiation zone approx?

time averaged energy current

$$\langle \vec{S}_{E1} \rangle = \frac{1}{\tau} \int_0^\tau dt\, \vec{S}_{E1}(\vec{r}, t) \qquad \tau \text{ is period } \tau = \frac{2\pi}{\omega}$$

$$\langle \cos^2(\ ) \rangle = \frac{1}{2}$$

$$= \frac{c}{8\pi} k^4 p_\omega^2 \frac{\sin^2\theta}{r^2} \hat{r}$$

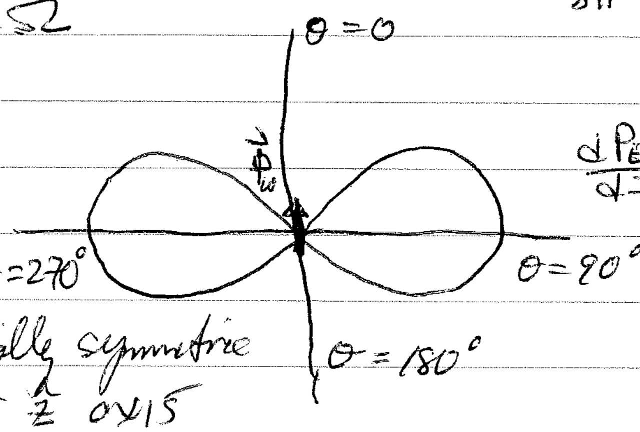
average energy flowing through an element of area at spherical angles θ, φ is

$$dP_{E1} = \hat{r} \cdot \langle \vec{S}_{E1} \rangle \underbrace{r^2 \sin\theta\, d\theta\, d\varphi}_{\substack{\text{area of surface element}\\ = r^2 d\Omega}} \qquad \Omega \text{ is solid angle}$$

$$= \hat{r} \cdot \langle \vec{S}_{E1} \rangle r^2 d\Omega$$

$$\frac{dP_{E1}}{d\Omega} = \hat{r} \cdot \langle \vec{S}_{E1} \rangle r^2 = \frac{c}{8\pi} k^4 p_\omega^2 \sin^2\theta \sim \omega^4 \sin^2\theta$$

polar plot



$\frac{dP_{E1}}{d\Omega} \sim \sin^2\theta$

most of power is directed outwards into plane $\perp \vec{p}_\omega$, i.e. peaked about $\theta = 90°$

rotationally symmetric about $\hat{z}$ axis

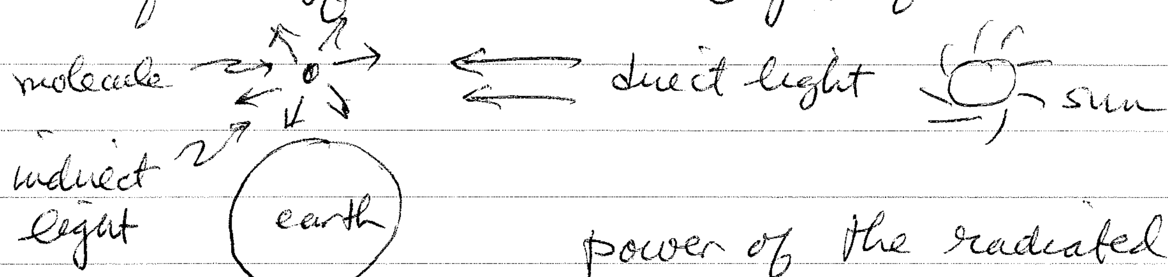
Total power radiated is

$$P_{E1} = \int \frac{dP_{E1}}{d\Omega}\, d\Omega = \frac{ck^4 p_\omega^2}{8\pi}\, 2\pi \underbrace{\int_0^\pi d\theta \sin\theta \sin^2\theta}_{4/3}$$

$$P_{E1} = \frac{ck^4 p_\omega^2}{3} = \frac{p_\omega^2 \omega^4}{3c^3} \sim \omega^4$$

<u>why the sky is blue – Lord Rayleigh</u>

when look up at sky, you are seeing the <u>indirect</u> light of the sun, which is the light emitted by the atoms and molecules of the atmosphere as they <u>oscillate</u> and so <u>radiate</u>, due to the electric field of the direct light from the sun



power of the radiated indirect light is $P \sim \omega^4 p_\omega^2$

$$\vec{p} = \alpha \vec{E} \qquad \alpha \sim \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

For molecules in atmosphere (N_2 etc) ω_0 is typically at a freq <u>higher</u> than visible spectrum. Therefore, in visible spectrum $\alpha \sim \frac{e^2}{m\omega_0^2}$ indep of ω.

$\Rightarrow$ power radiated is $P \sim \omega^4$

$P \sim \omega^4$ largest at high freq

Since light from sun is "white light" it has components of all freqs. Of these freqs, the higher ones are scattered the most & make up the indirect light we see.

Since blue is the largest ω in visible spectrum, the sky is blue!

When we look at sunrise or sunset, we are looking at the direct rays of the sun. Since these rays are least scattered at low ω ⇒ sunset and sunrise are red!

Magnetic dipole approx – Radiation Zone $kr \gg 1$

$$\vec{A}_{M1} = \frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\left(-\hat{r}\times\vec{m}_\omega\right)$$

$$\simeq ik\,\hat{r}\times\vec{m}_\omega\,\frac{e^{ikr}}{r} \quad \text{in RZ}$$

$$\vec{B}_{M1} = \vec{\nabla}\times\vec{A}_{M1} = \left(\vec{\nabla}e^{ikr}\right)\times\left(\frac{ik\hat{r}\times\vec{m}_\omega}{r}\right)$$

$$+\, e^{ikr}\,\underbrace{\vec{\nabla}\times\left(\frac{ik\hat{r}\times\vec{m}_\omega}{r}\right)}$$

will give terms of $o\left(\frac{1}{r^2}\right)$
so ignore in RZ approx

$$\boxed{\vec{B}_{M1} = -k^2\frac{e^{ikr}}{r}\,\hat{r}\times\left(\hat{r}\times\vec{m}_\omega\right)}$$

From Ampere's Law

$$\vec{E}_{M1} = \frac{i}{k}\vec{\nabla}\times\vec{B}_{M1} = -ik\left(\vec{\nabla}e^{ikr}\right)\times\left(\frac{\hat{r}\times[\hat{r}\times\vec{m}_\omega]}{r}\right)$$

$$-\, ik\,e^{ikr}\,\underbrace{\vec{\nabla}\times\left(\frac{\hat{r}\times[\hat{r}\times\vec{m}_\omega]}{r}\right)}$$

will give terms of $o\left(\frac{1}{r^2}\right)$
so ignore in RZ approx

$$\vec{E}_{M1} = k^2\frac{e^{ikr}}{r}\,\hat{r}\times\left(\hat{r}\times\left(\hat{r}\times\vec{m}_\omega\right)\right)$$

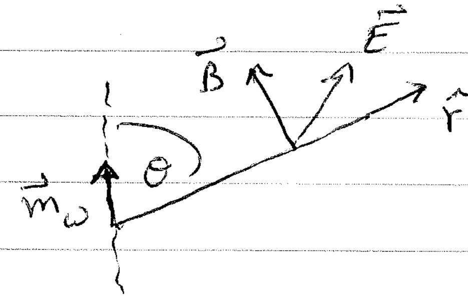
triple product rule

$$= k^2\frac{e^{ikr}}{r}\left\{\hat{r}\left[\hat{r}\cdot\left(\hat{r}\times\vec{m}_\omega\right)\right] - \left(\hat{r}\times\vec{m}_\omega\right)\left[\hat{r}\cdot\hat{r}\right]\right\}$$

$$\boxed{\vec{E}_{M1} = -\frac{k^2}{r}e^{ikr}\left(\hat{r}\times\vec{m}_\omega\right)}$$

If allign axes so that $\vec{m}_\omega = m_\omega \hat{z}$ then

$$\vec{E}_{M1} = m_\omega \frac{k^2}{r} e^{ikr} \sin\theta \, \hat{\varphi}$$



$$\vec{B}_{M1} = -m_\omega \frac{k^2}{r} e^{ikr} \sin\theta \, \hat{\theta}$$

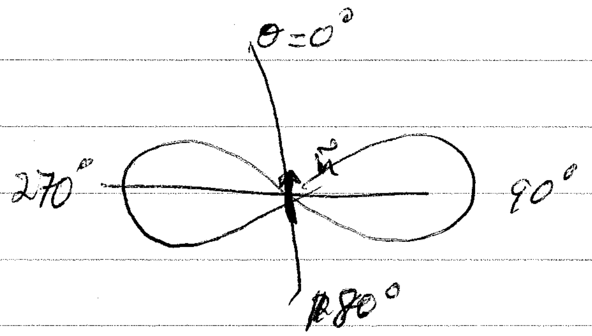
Poynting vector

$$\vec{S}_{M1} = \frac{c}{4\pi} \mathrm{Re}\{\vec{E}_{M1}\} \times \mathrm{Re}\{\vec{B}_{M1}\}$$

$$= \frac{c}{4\pi} \frac{k^4 m_\omega^2}{r^2} \cos^2(kr-\omega t) \sin^2\theta \, \hat{r}$$

$$\langle \vec{S}_{M1} \rangle = \frac{c}{8\pi} k^4 m_\omega^2 \frac{\sin^2\theta}{r^2} \hat{r}$$

$$\frac{dP_{M1}}{d\Omega} = \hat{r} \cdot \langle \vec{S}_{M1} \rangle r^2 = \frac{c}{8\pi} k^4 m_\omega^2 \sin^2\theta \sim \omega^4 \sin^2\theta$$



rotationally symmetric about $\hat{z}$ axis

$$P_{M1} = \int d\Omega \frac{dP_{M1}}{d\Omega} = \frac{ck^4}{3} m_\omega^2 = \frac{m_\omega^2 \omega^4}{3c^3}$$

$$\frac{P_{M1}}{P_{E1}} = \frac{m_\omega^2}{p_\omega^2} \sim \left(\frac{v}{c}\right)^2 \quad m_\omega \sim \frac{dj}{c} \sim dq \frac{v}{c}$$

$$p_\omega \sim dq$$

Electric Quadrupole radiation – radiation zone approx

$$\vec{A}_{E2} = \frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\left(\frac{-i\omega}{6c}\,\hat{r}\cdot\overleftrightarrow{Q}_\omega\right)$$

$$= -\frac{e^{ikr}}{r}\frac{k^3}{6}\,\hat{r}\cdot\overleftrightarrow{Q}_\omega \quad \text{in RZ approx}$$

$$\vec{B}_{E2} = \vec{\nabla}\times\vec{A}_{E2} = -(\vec{\nabla}e^{ikr})\times\left[\frac{k^2\,\hat{r}\cdot\overleftrightarrow{Q}_\omega}{6r}\right]$$

$$-\,e^{ikr}\underbrace{\vec{\nabla}\times\left[\frac{k^2\,\hat{r}\cdot\overleftrightarrow{Q}_\omega}{6r}\right]}_{O(\frac{1}{r^2})\text{ so ignore in RZ approx}}$$

$$\boxed{\vec{B}_{E2} = -ik^3\frac{e^{ikr}}{6r}\,\hat{r}\times\left(\hat{r}\cdot\overleftrightarrow{Q}_\omega\right)}$$

$$\vec{E}_{E2} = \frac{i}{k}\vec{\nabla}\times\vec{B}_{E2} = k^2(\vec{\nabla}e^{ikr})\times\left[\frac{\hat{r}\times(\hat{r}\cdot\overleftrightarrow{Q}_\omega)}{6r}\right]$$

$$+\,k^2e^{ikr}\underbrace{\vec{\nabla}\times\left[\frac{\hat{r}\times(\hat{r}\cdot\overleftrightarrow{Q}_\omega)}{6r}\right]}_{O(\frac{1}{r^2})\text{ so ignore in RZ approx}}$$

$$\vec{E}_{E2} = ik^3\frac{e^{ikr}}{6r}\,\underbrace{\hat{r}\times\left[\hat{r}\times(\hat{r}\cdot\overleftrightarrow{Q}_\omega)\right]}_{\text{triple product rule}}$$

$$= \frac{ik^3e^{ikr}}{6r}\left\{\hat{r}\left[\hat{r}\cdot(\hat{r}\cdot\overleftrightarrow{Q}_\omega)\right] - (\hat{r}\cdot\overleftrightarrow{Q}_\omega)[\hat{r}\cdot\hat{r}]\right\}$$

$$\boxed{\vec{E}_{E2} = \frac{ik^3e^{ikr}}{6r}\left\{\hat{r}\left(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r}\right) - \left(\hat{r}\cdot\overleftrightarrow{Q}_\omega\right)\right\}}$$

Poynting vector

$$\vec{S}_{E2} = \frac{+c}{4\pi} Re\{\vec{E}_{E2}\} \times Re\{\vec{B}_{E2}\}$$

$$= \frac{-k^6 \,(c/4\pi)}{36 r^2} \sin^2(kr-\omega t) \left\{ \hat{r}(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r}) - (\hat{r}\cdot\overleftrightarrow{Q}_\omega)\right\} \times \left[\hat{r}\times(\hat{r}\cdot\overleftrightarrow{Q}_\omega)\right]$$

$$= \frac{-k^6 \,(c/4\pi)}{36 r^2} \sin^2(kr-\omega t) \Big\{ \hat{r}\left[\hat{r}(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})\cdot(\hat{r}\overleftrightarrow{Q}_\omega)\right] - (\hat{r}\cdot\overleftrightarrow{Q}_\omega)\left[\hat{r}(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})\cdot\hat{r}\right]$$

$$- \hat{r}\left[(\hat{r}\cdot\overleftrightarrow{Q}_\omega)\cdot(\hat{r}\cdot\overleftrightarrow{Q}_\omega)\right] + (\hat{r}\cdot\overleftrightarrow{Q}_\omega)\left[(\hat{r}\cdot\overleftrightarrow{Q}_\omega)\cdot\hat{r}\right]\Big\}$$

$$= \frac{-k^6 \,(c/4\pi)}{36 r^2} \sin^2(kr-\omega t) \Big\{ (\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})^2\hat{r} - (\hat{r}\cdot\overleftrightarrow{Q}_\omega)(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})$$

$$- (\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})\hat{r} \quad (\hat{r}\cdot\overleftrightarrow{Q}_\omega)(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})\Big\}$$

$$\vec{S}_{E2} = \frac{-ck^6}{4\pi\, 36 r^2} \sin^2(kr-\omega t) \left\{ (\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})^2 - (\hat{r}\cdot\overleftrightarrow{Q}_\omega)^2 \right\} \hat{r}$$

$$\langle\vec{S}_{E2}\rangle = \frac{-ck^6}{4\pi\, 72 r^2} \left\{ (\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})^2 - (\hat{r}\cdot\overleftrightarrow{Q}_\omega)^2 \right\} \hat{r}$$

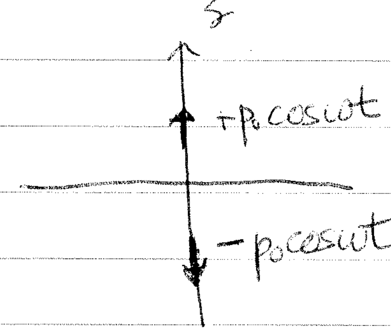
$$\frac{dP_{E2}}{d\Omega} = \hat{r}\cdot\langle\vec{S}_{E2}\rangle r^2 = \frac{ck^6}{4\pi 72} \left\{ (\hat{r}\cdot\overleftrightarrow{Q}_\omega)^2 - (\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})^2 \right\}$$

angular dependence of $\frac{dP_{E2}}{d\Omega}$ depends on specific form of the tensor $\overleftrightarrow{Q}_\omega$

For example: suppose $Q_{ij} = 0$ except for Q_{zz}

$$\Rightarrow \overleftrightarrow{Q}_\omega = Q_{zz}\,\hat{z}\hat{z}$$

ex



$$(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})^2 = (Q_{zz}\cos^2\theta)^2$$

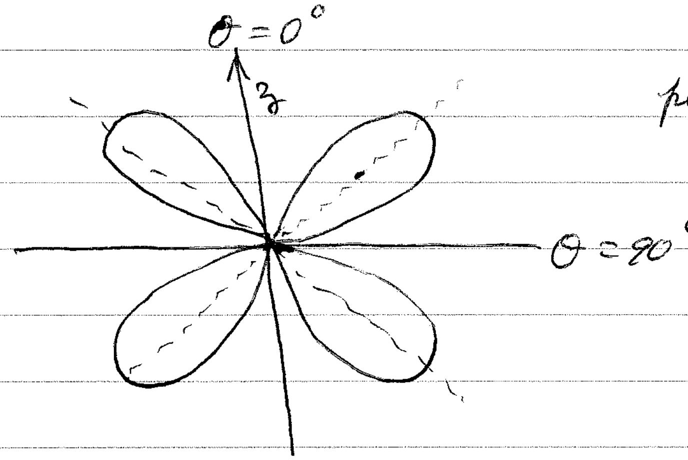
$$(\hat{r}\cdot\overleftrightarrow{Q}_\omega)^2 = Q_{zz}^2\cos^2\theta$$

$$\frac{dP_{E2}}{d\Omega} = \frac{ck^6}{4\pi 72}\,Q_{zz}^2\left[\cos^2\theta - \cos^4\theta\right]$$

$$= \frac{ck^6}{4\pi 72}\,Q_{zz}^2\cos^2\theta\sin^2\theta \qquad \cos\theta\sin\theta = \tfrac{1}{2}\sin 2\theta$$

$$= \frac{ck^6}{4\pi\, 288}\,Q_{z}^{22}\sin^2 2\theta$$

$\frac{dP_{E2}}{d\Omega}$



peak at 45°

rotationally invariant about $\hat{z}$ axis

$$\frac{P_{E2}}{P_{E1}} \sim \frac{k^6 Q^2}{k^4 p^2} \sim \frac{k^2 (qd^2)^2}{(qd)^2} \sim k^2 d^2 \sim \left(\frac{v}{c}\right)^2$$

$$P_{E2} \sim P_{M1}$$

For more general case, choose axes so that $\overleftrightarrow{Q}_\omega$ is diagonal - can always do this since $\overleftrightarrow{Q}_\omega$ is symmetric

$$(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r}) = \hat{r}\cdot\begin{pmatrix} Q_{xx} & 0 & 0 \\ 0 & Q_{yy} & 0 \\ 0 & 0 & Q_{zz}\end{pmatrix}\cdot\hat{r}$$

$$= \hat{r}\cdot\begin{pmatrix} Q_{xx}\sin\theta\cos\varphi \\ Q_{yy}\sin\theta\sin\varphi \\ Q_{zz}\cos\theta\end{pmatrix} = \begin{array}{l} Q_{xx}\sin^2\theta\cos^2\varphi \\ + Q_{yy}\sin^2\theta\sin^2\varphi \\ + Q_{zz}\cos^2\theta \end{array}$$

$$(\hat{r}\cdot\overleftrightarrow{Q}_\omega)^2 = Q_{xx}^2\sin^2\theta\cos^2\varphi + Q_{yy}^2\sin^2\theta\sin^2\varphi + Q_{zz}^2\cos^2\theta$$

$$\frac{dP_{E2}}{d\Omega} = \frac{k^6}{72}\Big\{ Q_{zz}^2(\cos^2\theta - \cos^4\theta) + Q_{xx}^2(\sin^2\theta\cos^2\varphi - \sin^4\theta\cos^4\varphi) + Q_{yy}^2(\sin^2\theta\sin^2\varphi - \sin^4\theta\sin^4\varphi)\Big\}$$

$$\frac{dP_{E2}}{d\Omega} = \frac{k^6}{72}\Big\{ Q_{zz}^2\cos^2\theta\sin^2\theta + Q_{xx}^2\sin^2\theta\cos^2\varphi(1-\sin^2\theta\cos^2\varphi) + Q_{yy}^2\sin^2\theta\sin^2\varphi(1-\sin^2\theta\sin^2\varphi)\Big\}$$

no special symmetries - varies with θ and φ

For arbitrary charge distributions – not pure harmonic

For $\vec{p}_\omega e^{-i\omega t}$ pure harmonic oscillation, we found the radiated fields in electric dipole approx are

$$\vec{E} = \vec{E}_\omega e^{-i\omega t}, \quad \vec{B} = \vec{B}_\omega e^{-i\omega t}$$

$$\vec{E}_\omega = -k^2 \frac{e^{ikr}}{r} \hat{r} \times (\hat{r} \times \vec{p}_\omega) = -\frac{\omega^2}{c^2} \frac{e^{i\omega r/c}}{r} \hat{r} \times (\hat{r} \times \vec{p}_\omega)$$

$$\vec{B}_\omega = k^2 \frac{e^{ikr}}{r} (\hat{r} \times \vec{p}_\omega) = \frac{\omega^2}{c^2} \frac{e^{i\omega r/c}}{r} (\hat{r} \times \vec{p}_\omega)$$

as $k = \frac{\omega}{c}$

For an arbitrarily time varying charge distribution with electric dipole moment

$$\vec{p}(t) = \int \frac{d\omega}{2\pi} \vec{p}_\omega e^{-i\omega t}$$

then solution for fields given by superposition

$$\vec{E}(\vec{r},t) = \int \frac{d\omega}{2\pi} \vec{E}_\omega e^{-i\omega t}$$

$$= -\int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-r/c)}}{r} \left(\frac{\omega^2}{c^2}\right) \hat{r} \times (\hat{r} \times \vec{p}_\omega)$$

$$= \frac{-1}{c^2 r} \hat{r} \times \left[\hat{r} \times \int \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \vec{p}_\omega \, \omega^2\right]$$

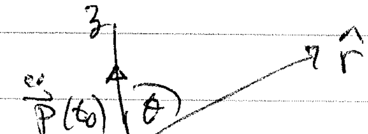
$$= \frac{1}{c^2 r} \hat{r} \times \left[\hat{r} \times \frac{\partial^2}{\partial t^2} \int \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \vec{p}_\omega\right]$$

$$\boxed{\vec{E}(\vec{r},t) = \frac{1}{c^2 r}\,\hat{r}\times\left[\hat{r}\times\ddot{\vec{p}}(t-r/c)\right]} \qquad \ddot{\vec{p}} = \frac{d^2\vec{p}}{dt^2}$$

define $t_0 \equiv t - r/c$ = "retarded time"

in spherical coords, if $\ddot{\vec{p}}(t_0)$ is along $\hat{z}$

$$\vec{E}(\vec{r},t) = \frac{\ddot{p}(t_0)}{c^2 r}\sin\theta\,\hat{\theta}$$



Similarly

$$\vec{B}(\vec{r},t) = \int \frac{d\omega}{2\pi}\,\vec{B}_\omega\, e^{-i\omega t}$$

$$= \int \frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-r/c)}}{r}\left(\frac{\omega^2}{c^2}\right)(\hat{r}\times\vec{p}_\omega)$$

$$= \frac{-1}{c^2 r}\,\hat{r}\times\frac{\partial^2}{\partial t^2}\int\frac{d\omega}{2\pi}\,e^{-i\omega(t-r/c)}\,\vec{p}_\omega$$

$$\boxed{\vec{B}(\vec{r},t) = \frac{-1}{c^2 r}\,\hat{r}\times\ddot{\vec{p}}(t_0)}$$

$$\vec{B}(\vec{r},t) = \frac{\ddot{p}(t_0)}{c^2 r}\sin\theta\,\hat{\varphi} \qquad \text{in spherical coords}$$

<u>Poynting vector</u>

$$\vec{S} = \frac{c}{4\pi}\,\vec{E}\times\vec{B} = \frac{c}{4\pi}\left(\frac{1}{c^2 r}\right)^2\left[\ddot{p}(t_0)\right]^2\sin^2\theta\,\hat{r}$$

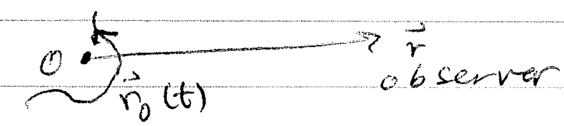
Total power radiated through a sphere of radius r is

$$\mathcal{P} = \oint da\, \hat{r} \cdot \vec{S} = 2\pi \int_0^{\pi} d\theta \sin\theta\, r^2\, \hat{r} \cdot \vec{S}$$

$$= \frac{[\ddot{p}(t_0)]^2}{2c^3} \underbrace{\int_0^{\pi} d\theta \sin^3\theta}_{4/3}$$

$$\boxed{\mathcal{P} = \frac{2[\ddot{p}(t_0)]^2}{3c^3}}$$

For a point charge moving along a trajectory $\vec{r}_0(t)$



$$\vec{p}(t) = q\vec{r}_0(t)$$

$$\ddot{\vec{p}}(t) = q\ddot{\vec{r}}_0(t) = q\vec{a}(t)$$

acceleration

$$\boxed{\mathcal{P} = \frac{2}{3} \frac{q^2 a^2(t_0)}{c^3}}$$

Larmor's formular

← total power passing through a sphere of radius r at time t is due to acceleration at retorded time $t_0 = t - r/c$

power radiated ~ (acceleration)^2

Larmor's formular above only holds in the non-relativestic limit since it is based on the electric dipole approx.