Special Relativity

1) Speed of light is constant in all inertial frames of reference

2) Physical laws must look the same in all inertial frames of reference — there is no experiment that can determine the "absolute" velocity of any inertial frame

$\Rightarrow$ If a flash of light goes off at the origin of some coord system, the outgoing wavefronts look spherical in all inertial frames.

Equation of wavefront is $r^2 - c^2 t^2 = 0$

$\Rightarrow$ (x, y, z, t) coords in one inertial frame K
(x', y', z', t') coords in another inertial frame K' that moves with velocity $\vec{v} = v\hat{x}$ with respect to K.

What is the transformation that relates coords in K' to coords in K

$y = y'$, $z = z'$

$\left(\begin{array}{l} \text{origins of } K \text{ and } K' \\ \text{coincide when } t = t' = 0 \end{array}\right.$

$\Rightarrow$ $c^2 t^2 - x^2 = c^2 t'^2 - x'^2$

$\Rightarrow$ $\dfrac{(ct + x)}{(ct' + x')} \dfrac{(ct - x)}{(ct' - x')} = 1$

Expect transformation to be __linear__

$\Rightarrow$ $ct' + x' = (ct + x) f$
$ct' - x' = (ct - x) f^{-1}$

for some constant f .   Write $f = e^{-y}$ y is rapidity

Solve for ct' and x' in terms of ct and x

$$ct' = ct\left(\frac{e^y + e^{-y}}{2}\right) - x\left(\frac{e^y - e^{-y}}{2}\right)$$

$$x' = -ct\left(\frac{e^y - e^{-y}}{2}\right) + x\left(\frac{e^y + e^{-y}}{2}\right)$$

$$ct' = ct\cosh y - x\sinh y$$
$$x' = -ct\sinh y + x\cosh y$$

meaning of parameter y

(at $x = 0$)

the origin of K has trajectory $x' = -vt'$ in K'

$\Rightarrow \dfrac{x'}{t'} = -v$

from transformation above, with $x = 0$, we get

$$\frac{x'}{ct'} = \frac{-ct\sinh y}{ct\cosh y} = -\tanh y$$

So $\dfrac{v}{c} = \tanh y$

$\Rightarrow \cosh y = \dfrac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \equiv \gamma$

$\sinh y = \left(\frac{v}{c}\right)\gamma$

Lorentz Transformation $\begin{cases} ct' = \gamma ct - \gamma\left(\frac{v}{c}\right)x \\ x' = -\gamma\left(\frac{v}{c}\right)ct + \gamma x \end{cases}$

Inverse transform obtained by taking $v \to -v$ in above

$$\begin{cases} ct = \gamma ct' + \gamma(\frac{v}{c}) x' \\ x = \gamma(\frac{v}{c}) ct' + \gamma x' \end{cases}$$

4 - vectors

<u>4 - position</u>: $x_\mu = (x_1, x_2, x_3, ict)$ $x_4 \equiv ict$

$x_\mu x_\mu \equiv \sum_{\mu=1}^{4} x_\mu^2 = r^2 - c^2 t^2$ Lorentz invariant scalar

— has same value in all

Lorentz transf is   interial frames

$x_1' = \gamma(x_1 + i(\frac{v}{c}) x_4)$

$x_2' = x_2$ linear transf, can be

$x_3' = x_3$ represented by a matrix

$x_4' = \gamma(x_4 - i(\frac{v}{c}) x_1)$

or $x_\mu' = a_{\mu\nu}(L) x_\nu$

$\underset{\uparrow}{\phantom{x_\mu' = a_{\mu\nu}(L) x_\nu}}$ matrix of Lorentz transformation L

$$a(L) = \begin{pmatrix} \gamma & 0 & 0 & i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix}$$

inverse: $x_\mu = a_{\mu\nu}(L^{-1}) x_\nu'$

$a_{\mu\nu}(L^{-1})$ is given by taking $v \to -v$ in $a_{\mu\nu}(L)$

we see $a_{\mu\nu}(L^{-1}) = a_{\nu\mu}(L)$

inverse = transpose

More generally

Since x_μ^2 is Lorentz invariant scalar,

$$x_\mu'^2 = a_{\mu\nu}(L) \, a_{\mu\lambda}(L) \, x_\nu x_\lambda = x_\lambda^2$$

$$\Rightarrow \quad a_{\mu\nu}(L) \, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow \quad a_{\nu\mu}^t(L) \, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow \quad a_{\mu\nu}^t = a_{\mu\nu}^{-1}(L) \quad \text{transpose} = \text{inverse}$$

$$a_{\mu\nu} \text{ is } 4 \times 4 \text{ orthogonal matrix}$$

If L_1 is a Lorentz transf from K to K'
$\quad L_2$ is a Lorentz transf from K' to K''
Then the Lorentz transf from K to K'' is given by the matrix

$$a(L_2 L_1) = a(L_2) \, a(L_1)$$

if $L_1 = L$ and $L_2 = L^{-1}$ so $L_2 L_1 = I$ identity

$$\Rightarrow \quad a^{-1}(L) = a(L^{-1})$$

$$dx_\mu = (dx_1, dx_2, dx_3, \, icdt)$$

$$-(dx_\mu)^2 \equiv c^2 ds^2 = c^2 dt^2 - dr^2 \quad \text{Lorentz invariant scalar}$$

$$ds^2 = dt^2 \left[1 - \frac{1}{c^2}\left(\frac{dx_1}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_2}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_3}{dt}\right)^2 \right]$$

$$ds^2 = \frac{dt^2}{\gamma^2}$$

$$\boxed{ds = \frac{dt}{\gamma}} \quad \text{proper time interval}$$

A 4-vector is any 4 numbers that transform under a Lorentz transformation the same way as does x_μ

4-velocity $\qquad u_\mu \equiv \dfrac{dx_\mu}{ds} \equiv \dot{x}_\mu$

$$= \gamma \frac{dx_\mu}{dt}$$

Space components $\vec{u} = \gamma \vec{v}$

$$u_4 = i c \gamma$$

$$u_\mu u_\mu = \gamma^2 v^2 - c^2 \gamma^2 = \gamma^2 (v^2 - c^2)$$

$$= \frac{v^2 - c^2}{1 - \frac{v^2}{c^2}} = -c^2$$

4-acceleration $\qquad \alpha_\mu \equiv \dfrac{du_\mu}{ds} = \gamma \dfrac{du_\mu}{dt}$

4-gradient $\qquad \dfrac{\partial}{\partial x_\mu} \equiv \left(\vec{\nabla}, -\dfrac{i}{c} \dfrac{\partial}{\partial t} \right)$

proof $\dfrac{\partial}{\partial x_\mu}$ is a 4-vector

$$\frac{\partial}{\partial x_\mu'} = \frac{\partial x_\lambda}{\partial x_\mu'} \frac{\partial}{\partial x_\lambda} \qquad\qquad \text{but} \quad \frac{\partial x_\lambda}{\partial x_\mu'} = a_{\lambda\mu}(L^{-1})$$
$$= a_{\mu\lambda}(L)$$
$$= a_{\mu\lambda}(L) \frac{\partial}{\partial x_\lambda}$$

So transforms same as x_μ

$$\left(\frac{\partial}{\partial x_\mu} \right)^2 = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \qquad \text{wave equation operator !}$$

inner products

If u_μ and v_μ are 4-vectors, then $u_\mu v_\mu$ is Lorentz invariant scalar
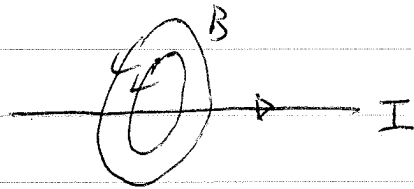
Electromagnetism

Clearly $\vec{E}$ & $\vec{B}$ must transform into each other
under Lorentz transf.

in inertial frame K	in frame K' moving with $\vec{v} \parallel$ to wire

stationary line charg λ

cylindrical outward
electric field
no B- field

moving line charge gives current
$\Rightarrow$ B circulating around wire
as well as outward radial E

Lorentz force

$$\vec{F} = q\vec{E} + q \frac{\vec{v}}{c} \times \vec{B}$$

what is the velocity $\vec{v}$ here? velocity with respect to
what inertial frame? clearly $\vec{E}$ and $\vec{B}$ must change
from one inertial frame to another if this force law
can make sense.

charge density

Consider charge ΔQ contained in a vol ΔV.
ΔQ is a Lorentz invariant scalar.

Consider the reference frame in which the charge
is instantaneously at rest. In this frame

$$\Delta Q = \overset{\circ}{\rho} \, \Delta \overset{\circ}{V}$$

$\overset{\circ}{\rho}$ is charge density in the rest frame

$\Delta \overset{\circ}{V}$ is volume in the rest frame

$\overset{\circ}{\rho}$ is Lorentz invariant by definition

Now transform to another frame moving with $\vec{v}$

with respect to rest frame

ΔQ remains the same

$$\Delta V = \frac{\Delta \overset{\circ}{V}}{\gamma} \qquad \text{volume contracts in direction} \parallel \text{to } \vec{v}$$

$$\rho = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q}{\Delta \overset{\circ}{V}} \gamma = \overset{\circ}{\rho} \gamma$$

Current density is $\quad \vec{J} = \rho \vec{v} = \gamma \vec{v} \cdot \frac{\rho}{\gamma} = \overset{\circ}{\rho} \vec{u}$

Define 4-current $\boxed{j_\mu = (\vec{J}, \, ic\rho)} = \overset{\circ}{\rho} (\vec{u}, \, ic\gamma)$

$$= \overset{\circ}{\rho} \, u_\mu$$

It is 4-vector since u_μ is 4-vector and $\overset{\circ}{\rho}$ is Lorentz invariant scalar.

charge conservation

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = \boxed{\frac{\partial j_\mu}{\partial x_\mu} = 0}$$

Equation for potentials in Lorentz gauge

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\frac{4\pi}{c}\vec{f}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -4\pi\rho$$

$$\frac{\partial^2}{\partial x_\mu^2} = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \text{ is Lorentz invariant operator}$$

4-potential $\boxed{A_\mu = (\vec{A}, \; i\phi)}$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A_\mu = \boxed{-\frac{4\pi}{c}j_\mu = \frac{\partial^2 A_\mu}{\partial x_\mu^2}}$$

Lorentz gauge condition is

$$\vec{\nabla}\cdot\vec{A} + \frac{\partial\phi}{c\,\partial t} = \frac{\partial A_\mu}{\partial x_\mu} = 0$$

Electric and magnetic fields

$$B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \qquad i,j,k \text{ cyclic permutation of } 1,2,3$$

$$E_i = -\frac{\partial\phi}{\partial x_i} - \frac{\partial A_i}{c\,\partial t} = i\left(\frac{\partial A_4}{\partial x_i} - \frac{\partial A_i}{\partial x_4}\right)$$

Define   field stress tensor

$$\boxed{F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = -F_{\nu\mu}}$$

$$= \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ & & & 0 \end{pmatrix}$$

"curl" of a 4-vector is a 4×4 anti symmetric 2nd rank tensor