Equation for potentials in Lorentz gauge

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\frac{4\pi}{c}\vec{j}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -4\pi\rho$$

$$\frac{\partial^2}{\partial x_\mu^2} = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$$ is Lorentz invariant operator

<u>4-potential</u> $$\boxed{A_\mu = (\vec{A}, \; i\phi)}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A_\mu = \boxed{-\frac{4\pi}{c}j_\mu = \frac{\partial^2 A_\mu}{\partial x_\lambda^2}}$$

Lorentz gauge condition is

$$\vec{\nabla}\cdot\vec{A} + \frac{\partial\phi}{c\,\partial t} = \frac{\partial A_\mu}{\partial x_\mu} = 0$$

<u>Electric and magnetic fields</u>

$$B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}$$ i, j, k cyclic permutation of $1, 2, 3$

$$E_i = -\frac{\partial\phi}{\partial x_i} - \frac{\partial A_i}{c\,\partial t} = i\left(\frac{\partial A_4}{\partial x_i} - \frac{\partial A_i}{\partial x_4}\right)$$

Define <u>field stress tensor</u>

$$\boxed{F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = -F_{\nu\mu}}$$

$$= \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix}$$

"curl" of a 4-vector is a 4x4 anti symmetric 2nd rank tensor

Inhomogeneous Maxwell's equations can be written in the form

$$\boxed{\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} j_\mu} \Rightarrow \left[\begin{array}{l}\vec{\nabla}\cdot\vec{E} = 4\pi\rho \\ \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}\end{array}\right]$$

$$= \frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) = \frac{\partial}{\partial x_\mu}\underbrace{\left(\frac{\partial A_\nu}{\partial x_\nu}\right)}_{=0} - \frac{\partial^2 A_\mu}{\partial x_\nu^2}$$

$$\Rightarrow -\frac{\partial^2 A_\mu}{\partial x_\nu^2} = \frac{4\pi}{c} j_\mu$$ agrees with previous equation for A_μ

transformation law for 2nd rank tensor $F_{\mu\nu}$

$$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu}$$ use $A'_\mu = a_{\mu\sigma} A_\sigma$, $\frac{\partial}{\partial x'_\mu} = a_{\mu\lambda}\frac{\partial}{\partial x_\lambda}$

$$= a_{\nu\lambda} a_{\mu\sigma} \frac{\partial A_\lambda}{\partial x_\sigma}$$

$$- a_{\mu\sigma} a_{\nu\lambda} \frac{\partial A_\sigma}{\partial x_\lambda}$$

$$F'_{\mu\nu} = a_{\mu\sigma} a_{\nu\lambda} F_{\sigma\lambda}$$ lets one find $\vec{E}'$ and $\vec{B}'$ if one knows $\vec{E}$ and $\vec{B}$

For n^{th} rank tensor

$$T'_{\mu_1\mu_2\cdots\mu_n} = a_{\mu_1\nu_1} a_{\mu_2\nu_2} \cdots a_{\mu_n\nu_n} T_{\nu_1\nu_2\cdots\nu_n}$$

$\frac{\partial F_{\mu\nu}}{\partial x_\nu}$ is a 4-vector: proof:

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\sigma} a_{\nu\lambda} a_{\nu\gamma} \frac{\partial F_{\sigma\lambda}}{\partial x_\gamma}$$

but $a_{\nu\lambda} = a^{-1}_{\lambda\nu}$ since inverse = transpose

$a_{\nu\lambda} a_{\nu\gamma} = a^{-1}_{\lambda\nu} a_{\nu\gamma} = \delta_{\lambda\gamma}$

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\sigma} \frac{\partial F_{\sigma\lambda}}{\partial x_\gamma} \delta_{\lambda\gamma} = a_{\mu\sigma} \frac{\partial F_{\sigma\lambda}}{\partial x_\lambda}$$ transforms like 4-vector

To write the <u>homogeneous Maxwell Equations</u>

Construct 3rd rank co-variant tensor

$$\boxed{G_{\mu\nu\lambda} = \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu}}$$

transforms as $G'_{\mu\nu\lambda} = a_{\mu\alpha} a_{\nu\beta} a_{\lambda\gamma} G_{\alpha\beta\gamma}$

in principle G has $4^3 = 64$ components

But can show that G is antisymmetric in exchange of any two indices

$$G_{\nu\mu\lambda} = \frac{\partial F_{\nu\mu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\nu}}{\partial x_\mu} + \frac{\partial F_{\mu\lambda}}{\partial x_\nu}$$

$$= -\frac{\partial F_{\mu\nu}}{\partial x_\lambda} - \frac{\partial F_{\nu\lambda}}{\partial x_\mu} - \frac{\partial F_{\lambda\mu}}{\partial x_\nu}$$ as F antisymmetric

$$= -G_{\mu\nu\lambda}$$

Also $G_{\mu\nu\lambda} = 0$ if any two indices are equal

$\Rightarrow$ only 4 independent components

$$G_{012}, \quad G_{013}, \quad G_{023}, \quad G_{123}$$

all other components either vanish or are $\pm$ one of the above.

the 4 homogeneous Maxwell Equations:

$$\vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

can be written as

$$\boxed{G_{\mu\nu\lambda} = 0}$$

to see, substitute in definition of G the definition of F

$$G_{\mu\nu\lambda} = \frac{\partial^2 A_\nu}{\partial x_\lambda \partial x_\mu} - \frac{\partial^2 A_\mu}{\partial x_\lambda \partial x_\nu} + \frac{\partial^2 A_\mu}{\partial x_\nu \partial x_\lambda} - \frac{\partial^2 A_\lambda}{\partial x_\nu \partial x_\mu} + \frac{\partial^2 A_\lambda}{\partial x_\mu \partial x_\nu} - \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\lambda}$$

all terms cancel in pairs

$$= 0$$

$$G_{123} = 0 \Rightarrow \vec{\nabla}\cdot\vec{B} = 0$$

$$G_{012} = -i\left[\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{c\,\partial t}\right]_z = 0$$ z component Faraday's law

Another way to write homogeneous Maxwell Equations

Define $\epsilon_{\mu\nu\lambda\sigma} = \begin{cases} +1 & \text{if } \mu\nu\lambda\sigma \text{ is even permutation of } 1234 \\ -1 & \text{if } \mu\nu\lambda\sigma \text{ is odd permutation of } 1234 \\ 0 & \text{otherwise} \end{cases}$

4-d LeviCivita symbol

Define

$$\tilde{F}_{\mu\nu} = \frac{1}{2i} \epsilon_{\mu\nu\lambda\sigma} F_{\lambda\sigma}$$

pseudo-tensor

has wrong sign under parity transf

$$= \begin{pmatrix} 0 & -E_3 & E_2 & -iB_1 \\ E_3 & 0 & -E_1 & -iB_2 \\ -E_2 & E_1 & 0 & -iB_3 \\ iB_1 & iB_2 & iB_3 & 0 \end{pmatrix}$$

$\frac{\partial \tilde{F}_{\mu\nu}}{\partial x_\nu} = 0$ gives homogeneous Maxwell equations

$$\left.\begin{aligned} \tfrac{1}{2} F_{\mu\nu} F_{\mu\nu} &= B^2 - E^2 \\ -\tfrac{1}{4} F_{\mu\nu} \tilde{F}_{\mu\nu} &= \vec{B}\cdot\vec{E} \end{aligned}\right\}$$ Lorentz invariant scalars

From $F'_{\mu\nu} = a_{\mu\sigma} a_{\nu\lambda} F_{\sigma\lambda}$ we can get Lorentz transf for $\vec{E}$ and $\vec{B}$

For a transformation from K to K' with K' moving with v along x, with respect to K,

$$E'_1 = E_1 \qquad B'_1 = B_1$$
$$E'_2 = \gamma\left(E_2 - \frac{v}{c} B_3\right) \qquad B'_2 = \gamma\left(B_2 + \frac{v}{c} E_3\right)$$
$$E'_3 = \gamma\left(E_3 + \frac{v}{c} B_2\right) \qquad B'_3 = \gamma\left(B_3 - \frac{v}{c} E_2\right)$$

Kinematics

"dot" is $\frac{d}{ds}$

4- momentum $\quad p_\mu = m\dot{x}_\mu = m u_\mu = (m\gamma\vec{v}, \ i m c \gamma)$

$$p_\mu^2 = m^2 u_\mu^2 = -m^2 c^2$$

4 - force $\quad K_\mu = (\vec{K}, i K_0)$ "Minkowski force"

Newton's 2nd law

$$m \frac{d^2 x_\mu}{ds^2} = K_\mu$$

$$\Rightarrow m \frac{d u_\mu}{ds} = \frac{d p_\mu}{ds} = K_\mu$$

$$p_\mu^2 = -m^2 c^2 \Rightarrow \frac{d}{ds}(p_\mu^2) = p_\mu \frac{d p_\mu}{ds} = p_\mu K_\mu = 0$$

$$\Rightarrow m\gamma \vec{v}\cdot\vec{K} - m c \gamma K_0 = 0 \quad \text{or}$$

$$K_0 = \frac{\vec{v}}{c}\cdot\vec{K}$$

Define the usual 3-force by

$$\frac{d\vec{p}}{dt} = \vec{F}$$

(we identify Newtonian momentum $\vec{p}$ with the space components of p_μ)

$$\frac{d\vec{p}}{ds} = \vec{K} \quad \text{and} \quad \frac{d\vec{p}}{ds} = \gamma \frac{d\vec{p}}{dt} = \gamma\vec{F} \quad \Rightarrow \quad \vec{K} = \gamma\vec{F}$$

$$K_0 = \gamma \frac{\vec{v}}{c} \cdot \vec{F}$$

Consider 4^{th}-component of Newton's eqn

$$m\frac{d}{ds}u_4 = m\frac{d}{ds}(ic\gamma) = iK_0 = i\gamma\frac{\vec{v}}{c}\cdot\vec{F}$$

$$d(m\gamma) = \gamma\frac{\vec{v}}{c^2}\cdot\vec{F}\,ds = \frac{dt\,\vec{v}\cdot\vec{F}}{c^2} = \frac{d\vec{r}\cdot\vec{F}}{c^2}$$

Work-energy theorem: $d(m\gamma c^2) = d\vec{r}\cdot\vec{F}$ = work done

$\Rightarrow$ $d(m\gamma c^2)$ is change in kinetic energy

$E = m\gamma c^2$ is relativistic kinetic energy

$$\boxed{p_\mu = \left(\vec{p}, \frac{iE}{c}\right) \qquad \begin{array}{l} \vec{p} = m\gamma\vec{v} \\ E = m\gamma c^2 \end{array}}$$

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \underset{\text{small } \frac{v}{c}}{\approx} mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) = \underbrace{mc^2}_{\text{rest mass energy}} + \underbrace{\frac{1}{2}mv^2}_{\text{non-rel kinetic energy}}$$

$\frac{dp_\mu}{ds} = K_\mu$ is therefore relativistic analog of Newton's 3rd law as well as law of conservation of energy

Lorentz force

$$\frac{dp_\mu}{ds} = K_\mu$$

what is the K_μ that represents the Lorentz force and how can we write it in ~~relativistic~~ Lorentz covariant way?

K_μ should depend on the fields $F_{\mu\nu}$ and the particles trajectory x_μ

as $\vec{v} \to 0$ $\quad \vec{K} = q\vec{E}$

K_μ can't depend directly on x_μ as should be indep of origin of coords. So can depend only on $\mathring{x}_\mu$, $\mathring{\mathring{x}}_\mu$, etc.

as $v \to 0$, K does not depend on the acceleration, so K does not depend on $\mathring{\mathring{x}}_\mu$

K_μ only depends on $F_{\mu\nu}$ and $\mathring{x}_\mu$
we need to form a 4-vector out of $F_{\mu\nu}$ and $\mathring{x}_\mu$ that is <u>linear</u> in the fields $F_{\mu\nu}$ and proportional to the charge q.

The only possibility is

$$q\, f(\mathring{x}_\mu^2)\, F_{\mu\nu} \mathring{x}_\nu$$

But $\mathring{x}_\mu^2 = -c^2$ is a constant. Choose $f(x_\mu^2) = \frac{1}{c}$

$$K_\mu = \frac{q}{c} F_{\mu\nu} \mathring{x}_\nu \quad \text{is only possibility}$$

This gives force

$$\vec{F} = \frac{1}{\gamma} \vec{K}$$

$$F_i = \frac{1}{\gamma} K_i = \frac{q}{\gamma c} \left(F_{ij} \mathring{x}_j + F_{i4} \mathring{x}_4 \right)$$

$$= \frac{q}{\gamma c} \left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) \mathring{x}_j + \frac{q}{\gamma c} (-iE_i)(ic\gamma)$$

$$= \frac{q}{\gamma c} \left[\varepsilon_{ijk} B_k \gamma v_j \right] + \frac{q}{\gamma c} E_i c \gamma$$

$$= q E_i + q \varepsilon_{ijk} \frac{v_j}{c} B_k$$

$$\vec{F} = q \vec{E} + q \frac{\vec{v}}{c} \times \vec{B}$$

Lorentz force is the same form in all inertial frames. No relativistic modification is needed.

Relativistic Larmor's formula

non-relativistic $\mathcal{P} = \frac{2}{3} \frac{q^2 [a(t_0)]^2}{c^3}$

Consider inertial frame in which charge is instantaneously at rest. Call this rest frame $\overset{\circ}{K}$.

power radiated in $\overset{\circ}{K}$ is $\overset{\circ}{P} = \frac{d\overset{\circ}{\mathcal{E}}(\overset{\circ}{t})}{d\overset{\circ}{t}}$

where $\overset{\circ}{\mathcal{E}}$ is energy radiated. In $\overset{\circ}{K}$, the momentum density $\overset{\circ}{\vec{\Pi}} = \frac{1}{4\pi c} \overset{\circ}{\vec{E}} \times \overset{\circ}{\vec{B}} \sim \hat{\overset{\circ}{r}}$ is in outward radial direction. Integrating over all directions, the radiated momentum vanishes $\overset{\circ}{\vec{p}} = 0$

energy-momentum is a 4-vector $(\overset{\circ}{\vec{p}}, \frac{i}{c}\overset{\circ}{\mathcal{E}})$

To get radiated energy in original frame K we can use Lorentz transf

$$\frac{\mathcal{E}}{c} = \gamma\left(\frac{\overset{\circ}{\mathcal{E}}}{c} - \frac{\vec{v}}{c} \cdot \overset{\circ}{\vec{p}}\right) \Rightarrow \mathcal{E} = \gamma \overset{\circ}{\mathcal{E}} \quad \text{as } \overset{\circ}{\vec{p}} = 0$$

and $dt = \gamma\, d\overset{\circ}{t}$ is time interval in K

($d\overset{\circ}{\vec{r}} = 0$ as charge stays at origin in $\overset{\circ}{K}$)

$$\text{So } \frac{d\mathcal{E}}{dt} = \frac{\gamma\, d\overset{\circ}{\mathcal{E}}}{\gamma\, d\overset{\circ}{t}} = \frac{d\overset{\circ}{\mathcal{E}}}{d\overset{\circ}{t}} \Rightarrow \mathcal{P} = \overset{\circ}{\mathcal{P}}$$

radiated power is Lorentz invariant!

in $\overset{\circ}{K}$ we can use non-relativistic Larmor's formula since $v = 0$. So

$$P = \frac{2}{3} \frac{q^2 \overset{\circ}{a}{}^2}{c^3} \qquad \overset{\circ}{a} \text{ is acceleration in } \overset{\circ}{K}$$

To write an expression with out explicitly making mention of frame $\overset{\circ}{K}$, we need to find a Lorentz invariant scalar that reduces to a^2 as $v \to 0$. Only choice is α_μ^2 the 4-acceleration $\alpha_\mu = \frac{du_\mu}{ds}$

$$\alpha_\mu = \frac{du_\mu}{ds} = \gamma \frac{du_\mu}{dt} = \gamma \frac{d}{dt}\left(\gamma \vec{v}, ic\gamma\right)$$

$$\vec{\alpha} = \gamma^2 \frac{d\vec{v}}{dt} + \gamma \vec{v} \frac{d\gamma}{dt}$$

$$\alpha_4 = ic\gamma \frac{d\gamma}{dt}$$

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(\frac{1}{\sqrt{1 - v^2/c^2}}\right) = \frac{\frac{\vec{v}}{c^2} \cdot \frac{d\vec{v}}{dt}}{(1 - v^2/c^2)^{3/2}} = \frac{1}{c^2}\gamma^3 \vec{v} \cdot \vec{a}$$

as $\vec{v} \to 0$, $\gamma \to 1$, $\frac{d\gamma}{dt} \to 0$, so $\left\{ \begin{array}{l} \vec{\alpha} \to \frac{d\vec{v}}{dt} = \vec{a} \\ \alpha_4 \to 0 \end{array} \right.$

$\alpha_\mu^2 \to |\vec{a}|^2$ as desired

<u>Relativistic Larmor's formula</u>

$$P = \frac{2}{3} \frac{q^2}{c^3} \alpha_\mu^2 = \frac{2}{3} \frac{q^2}{c^3} (\dot{u}_\mu)^2$$

$$\alpha_\mu = \left(\gamma^2 \frac{d\vec{v}}{dt} + \gamma \vec{v} \frac{d\gamma}{dt} , i c \gamma \frac{d\gamma}{dt}\right)$$

$$\frac{d\gamma}{dt} = \frac{1}{c^2} \gamma^3 \vec{v} \cdot \vec{a}$$

$$\alpha_\mu = \left(\gamma^2 \vec{a} + \gamma^4 \frac{1}{c^2} (\vec{v} \cdot \vec{a}) \vec{v} , \frac{i c \gamma^4 \vec{v} \cdot \vec{a}}{c^2} \right)$$

$$\alpha_\mu^2 = \gamma^4 a^2 + \gamma^8 \frac{(\vec{v} \cdot \vec{a})^2 v^2}{c^4} + \frac{2\gamma^6}{c^2} (\vec{v} \cdot \vec{a})^2 - \frac{\gamma^8 (\vec{v} \cdot \vec{a})^2}{c^2}$$

$$= \gamma^4 \left[a^2 + \gamma^4 \frac{(\vec{v} \cdot \vec{a})^2}{c^2} \left(\frac{v^2}{c^2} - 1 \right) + \frac{2\gamma}{c^2} (\vec{v} \cdot \vec{a})^2 \right]$$

$$= \gamma^4 \left[a^2 - \gamma^2 \frac{(\vec{v} \cdot \vec{a})^2}{c^2} + \frac{2\gamma^2 (\vec{v} \cdot \vec{a})^2}{c^2} \right]$$

$$\alpha_\mu^2 = \gamma^4 \left[a^2 + \gamma^2 \frac{(\vec{v} \cdot \vec{a})^2}{c^2} \right]$$

as $\vec{v} \to 0$, $\alpha_\mu^2 \to a^2$

$\alpha_\mu^2 = \mathring{a}^2$ Lorentz invariant

$\mathring{a}$ = acceleration in instantaneous rest frame

For a charge accelerating in linear motion, ~~where~~ $(\vec{v} \cdot \vec{a})^2 = v^2 a^2$

$$\alpha_\mu^2 = \gamma^4 a^2 \underbrace{\left(1 + \gamma^2 \frac{v^2}{c^2} \right)}_{=\gamma^2} = \gamma^6 a^2$$

$$P = \frac{2 a^2}{3 c^3} \gamma^6$$

For a charge in circular motion $(\vec{v} \cdot \vec{a}) = 0$

$$\alpha_\mu^2 = \gamma^4 a^2$$

$$P = \frac{2}{3} \frac{a^2}{c^3} \gamma^4$$