Since we know that the above wave equation describes electromagnetic waves, i.e., light, then

\[ \frac{k_5 k_3}{k_1} = \frac{1}{c^2} \]

we already had \( \frac{k_4 k_5}{k_1} = \frac{1}{c^2} \)

\[ \Rightarrow k_3 = k_4 \]

\[ \Rightarrow k_1 \text{ and } k_4 \text{ are arbitrary – they can be chosen to be anything, by adjusting the units of } B \text{ and } E. \text{ } k_3 \text{ and } k_5 \text{ are then fixed by } \frac{k_4 k_5}{k_1} = \frac{1}{c^2} \text{ implying } k_3 = k_4 \]

**Popular Systems of E&M units**

<table>
<thead>
<tr>
<th>System</th>
<th>( k_1 )</th>
<th>( k_2 = k_4 )</th>
<th>( k_5 )</th>
<th>( \varepsilon_0 \mu_0 = \frac{1}{c^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>MKS or SI</td>
<td>( \frac{1}{4\pi \varepsilon_0} )</td>
<td>1</td>
<td>( \frac{\mu_0}{4\pi} )</td>
<td>(\text{simplified})</td>
</tr>
<tr>
<td>Gaussian or CGS</td>
<td>1</td>
<td>( \frac{1}{c} )</td>
<td>( \frac{1}{c} )</td>
<td></td>
</tr>
<tr>
<td>Rationalized Gaussian</td>
<td>( \frac{1}{4\pi} )</td>
<td>( \frac{1}{c} )</td>
<td>( \frac{1}{4\pi c} )</td>
<td></td>
</tr>
</tbody>
</table>

In MKS, charges are measured in “coulombs”
current measured in “amps”
magnetic field measured in “tesla” = “weber/m²”
In CGS, charges are measured in "statcoulombs"  
current measured in "statamperes"  
magnetic field measured in "gauss"  
1 tesla = 10^4 gauss

We will use the CGS or Gaussian units.

\[ \mathbf{\nabla} \cdot \mathbf{E} = 4\pi \rho \]
\[ \mathbf{\nabla} \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \]
\[ \mathbf{\nabla} \cdot \mathbf{B} = 0 \]
\[ \mathbf{\nabla} \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \]

Maxwell's Equs.

\[ \mathbf{F} = q \left( \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) \]

Lorentz force

Physical content of Maxwell's equs.

1) Gauss Law for electric field — charge is source of E field. Field lines can begin and end at point charges.

2) Faraday's Law of Induction — time varying magnetic flux produces circulating E-field.

3) Gauss Law for magnetic fields — no magnetic monopoles. Magnetic field lines are continuous, they either close upon themselves or go off to infinity, they cannot begin nor end at any point.

4) Ampere's Law + Maxwell's Correction — electric current is a source for circulating B-field; so is a time varying E-field. Maxwell's correction is necessary to have charge conservation and to give electromagnetic waves.
Note:
\[ \nabla \cdot (\nabla \times \mathbf{E}) = 0 \quad \text{as div of curl always vanishes,} \]
then \((2) \Rightarrow -\frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{B}) = 0 \)

So if \( \nabla \cdot \mathbf{B} = 0 \) at \( t = 0 \), Eqn (2) requires that \( \nabla \cdot \mathbf{B} \)
remains zero for all time.

Similarly
\[ \nabla \cdot (\nabla \times \mathbf{B}) = 0 \]
then \((4) \Rightarrow \frac{4\pi}{c} \nabla \cdot \mathbf{\dot{J}} + \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) = 0 \)

Use charge continuity \( \nabla \cdot \mathbf{\dot{J}} = -\frac{\partial \rho}{\partial t} \) to get
\[-\frac{4\pi}{c} \frac{\partial \rho}{\partial t} + \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) = 0 \]
\[ \Rightarrow \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E} - 4\pi \rho) = 0 \]

So if \( \nabla \cdot \mathbf{E} = 4\pi \rho \) at \( t = 0 \), Eqn (4) requires that \( \nabla \cdot \mathbf{E} = 4\pi \rho \) for all time.

Then Equs (1) ad (3) can be viewed as
"initial conditions". If they are true for any
particular moment in time, then Equs (2) ad (4)
ensure that the remain true for all time.

Equis (2) ad (3) are also referred to as the homogeneous
Maxwell Equs - they involve only the fields \( \mathbf{E} + \mathbf{B} \) ad
not the sources \( \mathbf{J} \) and \( \mathbf{\dot{J}} \). Eqs (1) ad (4) are referred
to as the inhomogeneous Maxwell's Equs - they
involve the sources \( \rho \) and \( \mathbf{\dot{J}} \).
Electromagnetic Potentials & Gauge Invariance

\[ \nabla \cdot \vec{E} = 4\pi \rho \]
\[ \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \]
\[ \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{B} = \frac{4\pi \vec{J}}{c} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \]

Consider first statics.

**Electrostatics**

\[ \nabla \times \vec{E} = 0 \]

Since \(\frac{\partial \vec{B}}{\partial t} = 0\) for statics.

From vector calculus, we know that if the curl of a vector is zero everywhere, then we can always write that vector field as the gradient of some scalar function \(\phi\).

\[ \vec{E} = -\nabla \phi \Rightarrow \nabla \times \vec{E} = -\nabla \times (\nabla \phi) = 0 \]

\(\phi\) is electrostatic potential.

Gauss Law becomes

\[ \nabla \cdot \vec{E} = -\nabla \cdot (\nabla \phi) = -\nabla^2 \phi = 4\pi \rho \]

\[ \nabla^2 \phi = -4\pi \rho \]

Poisson's Equation

In regions where \(\rho = 0\), we have

\[ \nabla^2 \phi = 0 \]

Laplace's Equation
In our discussion of Coulomb's Law we saw that the electric field from a distribution of localized charges was

\[
\vec{E}(\vec{r}) = \int d^3r' \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = -\nabla \phi
\]

We therefore see that the solution to Poisson's equation for a localized charge distribution \( \rho \) (with \( \vec{E} = 0 \) as \( \vec{r} \to \infty \)) is

\[
\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}
\]

We will soon spend a fair amount of time learning new ways to solve \( \nabla^2 \phi = \rho \), both for arbitrary \( \rho \) where we want an approximate to the above integral (multipole expansion), and for cases where \( \phi \) or \( \nabla \phi \) are predetermined on the surfaces of specified regions of space, such as conducting surfaces (boundary value problems).
Magnetostatics

\[ \nabla \cdot \mathbf{B} = 0 \]

From vector calculus, we know that if the divergence of a vector field vanishes everywhere, then it can always be written as the curl of another vector field \( \mathbf{A} \)

\[ \mathbf{B} = \nabla \times \mathbf{A} \Rightarrow \nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0 \]

\( \mathbf{A} \) is the magnetic vector potential

This remains true in general—not just in magnetostatics.

Amperes law becomes

\[ \nabla \times \mathbf{B} = \frac{4\pi \mathbf{j}}{c} \quad \text{(in magnetostatics } \frac{\partial \mathbf{E}}{\partial t} = 0) \]

\[ \nabla \times (\nabla \times \mathbf{A}) = \frac{4\pi \mathbf{j}}{c} \]

\[ \Rightarrow \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \frac{4\pi \mathbf{j}}{c} \]

Magnetostatic gauge invariance

There are many possible vector potentials \( \mathbf{A} \) that result in the same \( \mathbf{B} \). If \( \mathbf{A} \) is such that \( \nabla \times \mathbf{A} = \mathbf{B} \), then \( \mathbf{A}' = \mathbf{A} + \nabla \chi \) also gives \( \nabla \times \mathbf{A}' = \mathbf{B} \), since \( \nabla \times \nabla \chi = 0 \) for any scalar function \( \chi(x) \).
Therefore we can always choose to represent \( \mathbf{B} \) by a vector potential \( \mathbf{A} \) such that \( \nabla \cdot \mathbf{A} = 0 \).

**Proof:** Suppose we had \( \mathbf{A} = \nabla \times \mathbf{A} \) for some \( \mathbf{A} \) with \( \nabla \cdot \mathbf{A} = D(\mathbf{r}) \neq 0 \). Construct an \( \mathbf{A}' = \mathbf{A} + \nabla \chi \) with \( \chi \) chosen as follows:

\[
\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} + \nabla^2 \chi = 0 \quad \Rightarrow \quad \nabla^2 \chi = -\nabla \cdot \mathbf{A} = D
\]

Solve for \( \chi \), for example:

\[
\chi(\mathbf{r}) = \frac{\int d^3 r' \, D(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|}
\]

We then have constructed an \( \mathbf{A}' \) such that

\[
\nabla \times \mathbf{A}' = \mathbf{B} \quad \text{and} \quad \nabla \cdot \mathbf{A}' = 0.
\]

This freedom to choose various \( \mathbf{A}' \)'s that give the same \( \mathbf{B} \) is called **gauge invariance**. Imposing a particular additional constraint on \( \mathbf{A} \) that removes this freedom is called **fixing the gauge**. The choice \( \nabla \cdot \mathbf{A} = 0 \) is usually known as the **Coulomb gauge** (or **Landau gauge**). Going from \( \mathbf{A} \) to \( \mathbf{A}' = \mathbf{A} + \nabla \chi \) is called making a **gauge transformation**.

"Working in the Coulomb gauge" with \( \nabla \cdot \mathbf{A} = 0 \), Ampère's law becomes

\[
\nabla^2 \mathbf{A} = \frac{-4\pi}{c} \mathbf{j}
\]

Poisson's Eq.

For a localized current density

\[
\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int d^3 r' \, \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}
\]
Back to dynamics

\[ \nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{B} = \nabla \times \mathbf{A} \]

remains true

But now instead of \( \nabla \times \mathbf{E} = 0 \) we have

\[ \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \]

\[ \Rightarrow \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \mathbf{A}) = 0 \]

\[ \Rightarrow \nabla \times (\mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}) = 0 \]

\[ \Rightarrow \text{there exists a scalar potential } \phi \text{ such that} \]

\[ \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi \quad \text{or} \quad \mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \]

Gauss's law for electric field now becomes

\[ \nabla \cdot \mathbf{E} = 4\pi \rho = -\nabla^2 \phi - \frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \mathbf{A}) = 4\pi \rho \]

\[ \nabla^2 \phi + \frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \mathbf{A}) = -4\pi \rho \]

Gauss law in terms of electromagnetic potentials

Amperes's law becomes

\[ \nabla \times \mathbf{B} = \frac{4\pi \mathbf{j}}{c} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \]

\[ \nabla \times (\nabla \times \mathbf{A}) = \frac{4\pi \mathbf{j}}{c} + \frac{1}{c} \frac{\partial}{\partial t} \nabla \phi - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} \]
\[- \nabla^2 \vec{A} + \nabla (\nabla \cdot \vec{A}) = \frac{4\pi}{c^2} \frac{\phi}{t} - \frac{1}{c^2} \frac{\partial}{\partial t} \left( \nabla \cdot \vec{A} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) \]

\[- \nabla^2 \vec{A} + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c^2} \frac{\phi}{t} - \nabla (\nabla \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t}) \]

**Gauge invariance**

As before, we can always construct \( \vec{A}' = \vec{A} + \nabla \chi \)
for any scalar function \( \chi \), that gives the same \( \vec{B} \).

But since \( \vec{A} \) now also enters expression for \( \vec{E} \),
we need to make sure that if we change \( \vec{A} \) to \( \vec{A}' \), we
must make some corresponding change \( \phi \) to \( \phi' \) so that
\( \vec{E} \) does not change.

\[
\begin{bmatrix}
\vec{A}' = \vec{A} + \nabla \chi \\
\phi' = \phi - \frac{1}{c} \frac{\partial \chi}{\partial t}
\end{bmatrix} \quad \text{Gauge transformation}
\]

For any scalar \( \chi \), the above \( \vec{A}' \) and \( \phi' \) give the
same values of \( \vec{E} \) as \( \vec{A} \) and \( \phi \).

**Proof:**

\[\nabla \times \vec{A}' = \nabla \times \vec{A} + \nabla \times \nabla \chi = \nabla \times \vec{A} = \vec{B} \]

\[(- \nabla \phi' - \frac{1}{c} \frac{\partial \vec{A}'}{\partial t}) = - \nabla \phi + \frac{1}{c} \nabla \cdot \nabla \chi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \frac{1}{c^2} \frac{\partial \phi}{\partial t} \]

\[= \left( - \nabla \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = \vec{E} \]

As before, we can fix the gauge by imposing some additional constraint on \( \vec{A} \) and \( \phi \). There are two popular choices:
1) Lorentz Gauge

Gauge constraint: require \( \frac{1}{c} \frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0 \)

Then Gauss' law becomes

\( \nabla^2 \phi + \frac{1}{c^2} \frac{\partial}{\partial t} \left( \frac{1}{c} \frac{\partial \phi}{\partial t} \right) = -4\pi \rho \)

\[ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi \rho \]

Amperle's law becomes

\[ -\nabla^2 \vec{A} + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c} \vec{j} - \nabla \left( \vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right) \]

\[ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \vec{j} \]

The combination \( -\nabla^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \equiv \Box \)

is the wave equation operator.

In Lorentz gauge, \( \vec{A} \) and \( \phi \) satisfy the inhomogeneous wave equations:

\[ \Box^2 \vec{A} = \frac{4\pi}{c} \vec{j} \]

\[ \Box^2 \phi = 4\pi \rho \]

when \( \vec{j} = 0, \rho = 0 \) electromagnetic waves are solution!
Note: Lorentz gauge condition does not uniquely determine $\mathbf{A}$ and $\phi$. If one constructs $A' = A + \mathbf{v} \times x$ and $\phi' = \phi - \frac{1}{c} \frac{\partial x}{\partial t}$, then $A'$ and $\phi'$ will also be in Lorentz gauge as long as $\mathbf{v} \cdot \mathbf{x} = 0$ (proof left to reader).

2) **Coulomb Gauge**

Gauge condition: require $\nabla \cdot \mathbf{A} = 0$

If $A$ is in the Coulomb gauge, then $A' = A + \mathbf{v} \times x$ will also be in Coulomb gauge provided $\nabla^2 \mathbf{x} = 0$.

Then Gauss' law becomes

$$\nabla^2 \phi + \frac{1}{c^2} \frac{\partial}{\partial t} (\mathbf{v} \cdot \mathbf{A}) = -4\pi \rho$$

$$\Rightarrow \nabla^2 \phi = -4\pi \rho$$ same as electrostatics!

$$\Rightarrow \phi(\mathbf{r}) = \int d^3 r' \frac{\rho(\mathbf{r}', t)}{\left| \mathbf{r} - \mathbf{r}' \right|}$$

no matter what motion the source $\rho(\mathbf{r}, t)$ has! $\phi$ is given by the instantaneous Coulomb potential even though electromagnetic fields have a finite velocity of propagation $c$. 

Amperes Law becomes:

\[- \nabla^2 A + \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = \frac{\mu_0}{c} \vec{J} - \nabla (\nabla \cdot A + \frac{\partial \phi}{\partial t})\]

\[\Rightarrow \nabla^2 A = \frac{\mu_0}{c} \vec{J} - \frac{1}{c} \nabla \left( \frac{\partial \phi}{\partial t} \right)\]

where \(\nabla \left( \frac{\partial \phi}{\partial t} \right) = \nabla \left[ \int d^3r' \frac{\partial \phi}{\partial t} \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) \right]\]

\[= - \nabla \left( \int d^3r' \frac{\nabla' \cdot \vec{J}(\vec{r}, t)}{|\vec{r} - \vec{r}'|} \right) \quad \text{by continuity eqn.}\]

To see the meaning of this term, recall – any vector function \(\vec{J}\) can be written as the sum of a curlfree and a divergenceless part:

\[\vec{J} = \vec{J}_{II} + \vec{J}_{\perp} \quad \text{where} \quad \nabla \times \vec{J}_{II} = 0 \quad \text{curlfree} \quad \nabla \cdot \vec{J}_{\perp} = 0 \quad \text{divergenceless}\]

where:

\[\vec{J}_{II} (\vec{r}) = -\frac{1}{4\pi} \nabla \int d^3r' \nabla' \cdot \vec{J}(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} \quad \text{longitudinal part}\]

\[\vec{J}_{\perp} (\vec{r}) = \frac{1}{4\pi} \nabla \times \int d^3r' \frac{\nabla' \times \vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad \text{transverse part}\]

So \(\nabla \left( \frac{\partial \phi}{\partial t} \right) = 4\pi \vec{J}_{II}\), and

\[\nabla^2 A = \frac{\mu_0}{c} \vec{J} - \frac{4\pi}{c} \vec{J}_{II} = \frac{4\pi}{c} \vec{J}_{\perp}\]