Returning to Ampere's law we see that the term

$$\vec{\nabla}\left(\frac{\partial \phi}{\partial t}\right) = -\vec{\nabla} \int d^3r' \left[\frac{\vec{\nabla}' \cdot \vec{j}(r',t)}{|\vec{r}-\vec{r}'|}\right]$$

$$= 4\pi \vec{j}_{\parallel}(\vec{r},t)$$

So Amperes law becomes

$$\Box^2 \vec{A} = \frac{4\pi}{c}\vec{j} - \frac{4\pi}{c}\vec{j}_{\parallel}$$

$$\boxed{\Box^2 \vec{A} = \frac{4\pi}{c}\vec{j}_{\perp}}$$

In Coulomb gauge, only the transverse part of $\vec{j}$ serves as a source for $\vec{A}$.

$\vec{A}$ describes the <u>transverse</u> modes, ie the EM radiation (recall in EM waves, the fields are always $\perp$ direction of propagation)

ϕ describes the <u>longitudinal</u> modes

Coulomb gauge is <u>not</u> Lorentz invariant - if $\vec{\nabla}\cdot\vec{A} = 0$ in one inertial reference frame, in general $\vec{\nabla}\cdot\vec{A} \neq 0$ in another.

In Coulomb gauge, if $\rho = 0$, then $\phi = 0$ and

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

Transverse + Longitudinal Parts of vector functions

To prove the preceeding claim, $\vec{j} = \vec{j}_{\parallel} + \vec{j}_{\perp}$, where $\vec{\nabla}\times\vec{j}_{\parallel} = 0$ and $\vec{\nabla}\cdot\vec{j}_{\perp} = 0$, we first digress to prove Helmholtz Theorem.

Helmholtz Theorem: For a vector function $\vec{f}(\vec{r})$ if one knows the divergence and curl of $\vec{f}$ then one can ~~uniquely~~ uniquely determine $\vec{f}$ itself.

That is, if

$$\vec{\nabla}\cdot\vec{f} = 4\pi D(\vec{r})$$ where $D(\vec{r})$ is a known scalar function

$$\vec{\nabla}\times\vec{f} = 4\pi \vec{C}(\vec{r})$$ where $\vec{C}(\vec{r})$ is a known vector function.

~~Then one can solve for~~

And if well defined boundary conditions on $\vec{f}$ are known (here we will assume $\vec{f}(\vec{r}) \to 0$ as $\vec{r} \to \infty$) Then there is a unique solution for $\vec{f}(\vec{r})$.

We prove this by construction!

Assume a solution of the form

$$\vec{f} = -\vec{\nabla}\varphi + \vec{\nabla}\times\vec{W}$$ where φ is a scalar and $\vec{W}$ a vector

Now we show that we can find such a solution

First consider

$$\vec{\nabla}\cdot\vec{f} = -\nabla^2\varphi + \vec{\nabla}\cdot(\vec{\nabla}\times\vec{W}) = -\nabla^2\varphi + 0 = 4\pi D(\vec{r})$$

So $-\nabla^2\varphi = 4\pi D(\vec{r})$ This is just Poisson's eqn we saw in electrostatics

Solution when $\varphi(\vec{r})\to 0$ as $\vec{r}\to\infty$ is given by

$$\boxed{\varphi(\vec{r}) = \int d^3r' \, \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}$$

Coulomb-like integral solution

Now consider

$$\vec{\nabla}\times\vec{f} = -\vec{\nabla}\times\vec{\nabla}\varphi + \vec{\nabla}\times(\vec{\nabla}\times\vec{W}) = 0 - \nabla^2\vec{W} + \vec{\nabla}(\vec{\nabla}\cdot\vec{W})$$
$$= 4\pi\vec{C}(\vec{r})$$

Choose a gauge in which $\vec{\nabla}\cdot\vec{W} = 0$ (just like Coulomb gauge in magnetostatics)

Then $-\nabla^2\vec{W} = 4\pi\vec{C}(\vec{r})$

$$\boxed{\vec{W}(\vec{r}) = \int d^3r' \, \frac{\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}$$

just like solution for vector pot $\vec{A}$ in magnetostatics

So we have constructed a solution

$$f(\vec{r}) = -\vec{\nabla}\varphi + \vec{\nabla}\times\vec{W}$$

$$= -\vec{\nabla}\int d^3r' \, \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \vec{\nabla}\times\int d^3r' \, \frac{\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

where $\vec{\nabla}\cdot\vec{f} = 4\pi D$ and $\vec{\nabla}\times\vec{f} = 4\pi\vec{C}$

Note: For above solution to be well defined, the integrals must converge. They will converge if the "sources" $D(\vec{r})$ and $\vec{C}(\vec{r})$ are sufficiently "localized" in space, ie $D(\vec{r}) \to 0$, $\vec{C}(\vec{r}) \to 0$ sufficiently fast as $\vec{r} \to \infty$.

Now we show that the above solution is <u>unique</u>.

Suppose there was another solution $\vec{g}$ such that

$$\vec{\nabla} \cdot \vec{g} = 4\pi D \quad \text{and} \quad \vec{\nabla} \times \vec{g} = 4\pi \vec{C}$$

Consider $\vec{h} \equiv \vec{f} - \vec{g}$ then

$$\vec{\nabla} \cdot \vec{h} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{h} = 0$$

Can show that only such $\vec{h}$ that also has $\vec{h}(\vec{r}) \to 0$ as $\vec{r} \to \infty$ is $\vec{h} \equiv 0$, so $\vec{g} = \vec{f}$ and solution is unique.

As a consequence of Helmholtz theorem we have also shown the following

① Any vector function $\vec{f}$ can be written in terms of a scalar and vector potential

$$\vec{f} = -\vec{\nabla}\varphi + \vec{\nabla} \times \vec{W}$$

or equivalently

(2) Any vector function $\vec{f}$ can be written in terms of a curl free and a divergenceless part

$$\vec{f} = \vec{f}_{\parallel} + \vec{f}_{\perp} \qquad \text{where} \quad \vec{\nabla}\times\vec{f}_{\parallel} = 0 \quad \text{curlfree}$$
$$\vec{\nabla}\cdot\vec{f}_{\perp} = 0 \quad \text{divergenceless}$$

where
$$\begin{cases} \vec{f}_{\parallel}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r}) = -\vec{\nabla}\displaystyle\int \frac{d^3r'}{4\pi}\,\frac{[\vec{\nabla}'\cdot\vec{f}(\vec{r}')]}{|\vec{r}-\vec{r}'|} \\ \vec{f}_{\perp}(\vec{r}) = \vec{\nabla}\times\vec{W}(\vec{r}) = \vec{\nabla}\times\displaystyle\int \frac{d^3r'}{4\pi}\,\frac{[\vec{\nabla}'\times\vec{f}(\vec{r}')]}{|\vec{r}-\vec{r}'|} \end{cases}$$

where in above we used $\vec{D}(\vec{r}') = \frac{1}{4\pi}\vec{\nabla}'\cdot f(\vec{r}')$

$$\vec{C}(\vec{r}') = \frac{1}{4\pi}\vec{\nabla}'\times\vec{f}(\vec{r}')$$

~~where~~ $\vec{f}_{\parallel}$ is called the longitudinal part of $\vec{f}$

$\vec{f}_{\perp}$ is called the transverse part of $\vec{f}$

To understand the reason for these names, we need to consider the Fourier transforms

Above can be generalized to situations where $\vec{f}$ satisfies other boundary conditions, ie say has a specified value on a given boundary surface. One first replaces $\frac{1}{|\vec{r}-\vec{r}'|}$ by the appropriate Greens function — see more to come!

Discussion regarding Fourier transforms

$$\vec{f}(\vec{r}) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, \vec{f}(\vec{k}) \qquad \text{Fourier transf}$$

$$\vec{f}(\vec{k}) = \int_{-\infty}^{\infty} d^3r\, e^{-i\vec{k}\cdot\vec{r}}\, \vec{f}(\vec{r}) \qquad \text{inverse transf}$$

Some special cases well worth remembering

① Transform of Dirac function

$$\delta_{\vec{r}_0}(\vec{k}) \equiv \int d^3r\, e^{-i\vec{k}\cdot\vec{r}}\, \delta(\vec{r}-\vec{r}_0) = e^{-i\vec{k}\cdot\vec{r}_0}$$

$$\Rightarrow \quad \delta(\vec{r}-\vec{r}_0) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, \delta_{\vec{r}_0}(\vec{k})$$

$$\boxed{\delta(\vec{r}-\vec{r}_0) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}_0)}}$$

or letting $\vec{r} \leftrightarrow \vec{k}$ in the above

$$\delta(\vec{k}-\vec{k}_0) = \int \frac{d^3r}{(2\pi)^3}\, e^{i\vec{r}\cdot(\vec{k}-\vec{k}_0)}$$

② Transform of Coulomb potential $\frac{1}{|\vec{r}-\vec{r}'|}$

We know

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}')$$

Suppose $f(\vec{k}) \equiv \int d^3r\, e^{-i\vec{k}\cdot\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|}$ is the

Fourier transf of $\frac{1}{|\vec{r}-\vec{r}'|}$

Substitute
$$\begin{cases} \dfrac{1}{|\vec{r}-\vec{r}'|} = \displaystyle\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, f(\vec{k}) \\[2ex] \delta(\vec{r}-\vec{r}') = \displaystyle\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \end{cases}$$

into above Poisson equation

$$\nabla^2 \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, f(\vec{k}) = -4\pi \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

operates only on $\vec{r}$
so move inside integral

$$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = \vec{\nabla}\cdot\left(\vec{\nabla} e^{i\vec{k}\cdot\vec{r}}\right)$$

① $$\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = \sum_{i=1}^{3} \hat{x}_i \frac{\partial}{\partial x_i} e^{i\vec{k}\cdot\vec{r}} = \sum_{i=1}^{3} \hat{x}_i\, i k_i\, e^{i\vec{k}\cdot\vec{r}}$$

$$= i\vec{k}\, e^{i\vec{k}\cdot\vec{r}} \qquad \text{where } \hat{x}_1, \hat{x}_2, \hat{x}_3 = \hat{x}, \hat{y}, \hat{z}$$

② $$\vec{\nabla}\cdot\left(i\vec{k} e^{i\vec{k}\cdot\vec{r}}\right) = (i\vec{k})\cdot(i\vec{k})\, e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

so $$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

Poisson eqn then gives

$$\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} (-k^2) f(\vec{k}) = -4\pi \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, e^{-i\vec{k}\cdot\vec{r}'}$$

$$\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} \left[-k^2 f(\vec{k})\right] = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} \left[-4\pi e^{-i\vec{k}\cdot\vec{r}'}\right]$$

As is true for Fourier series, so it is true for Fourier transforms: If two functions are equal, then their Fourier transforms are equal.

$$\Rightarrow \quad -k^2 f(\vec{k}) = -4\pi e^{-i\vec{k}\cdot\vec{r}'}$$

$$\boxed{f(\vec{k}) = \frac{4\pi}{k^2} e^{-i\vec{k}\cdot\vec{r}'}}$$

$\Rightarrow$ is the Fourier transform of $\dfrac{1}{|\vec{r}-\vec{r}'|}$

Electrostatic

$-\nabla^2\phi = 4\pi\rho$ with $\vec{E} = -\vec{\nabla}\phi$ (statics only)

physical meaning of the potential ϕ

work done to move a test charge δq from $\vec{r}_1$ to $\vec{r}_2$ in presence of an electric field $\vec{E}$ is

$$W_{12} = \int_{\vec{r}_1}^{\vec{r}_2} d\vec{l}\cdot\vec{F}$$

where $\vec{F}$ is the force required to move the charge. Since $\vec{E}$ exerts a force $\delta q\,\vec{E}$ on the charge, $\vec{F}$ must counter-balance this electric force so we can move the charge quasi statically $\Rightarrow \vec{F} = -\delta q\,\vec{E}$

$$W_{12} = -\delta q\int_{\vec{r}_1}^{\vec{r}_2} d\vec{l}\cdot\vec{E} = \delta q\int_{\vec{r}_1}^{\vec{r}_2} d\vec{l}\cdot\vec{\nabla}\phi = \delta q\left[\phi(\vec{r}_2) - \phi(\vec{r}_1)\right]$$

$$\phi(\vec{r}_2) - \phi(\vec{r}_1) = \frac{W_{12}}{\delta q}$$

difference in potential between two points is the work per unit charge to move a test charge between the two points

Green's Functions – part I

$-\nabla^2\phi = 4\pi\rho$

We already know that for a point charge q at position $\vec{r}'$, ie $\rho(\vec{r}) = q\,\delta(\vec{r}-\vec{r}')$, the solution to the above is

$\phi(\vec{r}) = \dfrac{q}{|\vec{r}-\vec{r}'|}$ ie $-\nabla^2\left(\dfrac{1}{|\vec{r}-\vec{r}'|}\right) = 4\pi\delta(\vec{r}-\vec{r}')$

We call the special solution for a point source the Green function for the differential operator

$-\nabla^2 G(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$

$G(\vec{r},\vec{r}')$ gives the potential at position $\vec{r}$ due to a unit source at position $\vec{r}'$

Generally, one also has to specify a desired boundary condition for the Green function on the boundary of the system.

For the Coulomb solution for a point charge the implicit boundary condition is that the potential vanish infinitely far from the charge

$G(\vec{r},\vec{r}') \to 0$ as $|\vec{r}-\vec{r}'| \to \infty$

boundary of the system is taken to infinity

If one knows the Green's function, then one can find the solution for any distribution of sources $\rho(\vec{r})$

$$\phi(\vec{r}) = \int d^3r' \, G(\vec{r},\vec{r}\,') \, \rho(\vec{r}\,')$$

<u>proof:</u> $-\nabla^2\phi = \int d^3r' \left[-\nabla^2 G(\vec{r},\vec{r}\,')\right] \rho(\vec{r}\,')$

$$= \int d^3r' \left[4\pi\,\delta(\vec{r}-\vec{r}\,')\right] \rho(\vec{r}\,')$$

$$= 4\pi \rho(\vec{r})$$

We will return to concept of Greens function when we discuss solution of Poisson's eqn in a finite volume

We will also see Greens functions again when we discuss solution of the inhomogeneous wave equation.

The Coulomb problem as a boundary value problem

Consider a conducting sphere of radius R with net charge q (as $R \to 0$ we get a point charge). What is $\phi(\vec{r})$? What is $E(\vec{r})$?

Review: Properties of conductors in electrostatics

1) $\vec{E} = 0$ inside conductor – if $\vec{E} \neq 0$ then a current $\vec{j} = \sigma \vec{E}$ flows and it is not statics (σ is conductivity)
2) $\rho = 0$ inside conductor – if $\vec{E} = 0$ inside, then $\vec{\nabla} \cdot \vec{E} = 4\pi\rho = 0$
3) Any net charge on the conductor must lie on the surface – follows from (2)
4) $\phi =$ constant throughout conductor – if $\vec{E} = 0$ then $\vec{E} = -\vec{\nabla}\phi \Rightarrow \phi$ is constant
5) Just outside the conductor, $\vec{E}$ is $\perp$ to surface.
   – If $\vec{E}$ has a component $\parallel$ to surface then it exerts a force on electrons at the surface leading to a surface current – so would not be static

For conducting sphere, $\rho = 0$ for $r > R$ and $r < R$ all charge is on the surface $\Rightarrow \nabla^2 \phi = 0$ for $\begin{cases} r > R \\ r < R \end{cases}$

spherical symmetry $\Rightarrow$ expect spherically symmetric solution

$\Rightarrow \phi(\vec{r})$ depends only on $r = |\vec{r}|$