Green's function - part II

Greens 2nd identity

\[ \frac{1}{V} \int d^3r' \left( \phi \nabla'^2 - 4 \nabla'^2 \phi \right) = \oint_{\partial S'} \left( \phi \frac{\nabla G}{\partial m'} - 4 \frac{\partial \phi}{\partial m'} \right) \]

Apply above with \( \phi(r) \) electrostatic potential with \( \nabla^2 \phi = -4\pi \rho(r) \)

\( G(r, r') = G(r, r') \) the Green function satisfying

\[ \nabla'^2 G(r, r') = -4\pi \delta(r-r') \]

We saw one solution of above is

\[ G(r, r') = \frac{1}{|r-r'|} \]

but a more general solution is

\[ G(r, r') = \frac{1}{|r-r'|} + F(r, r') \]

where \( \nabla'^2 F(r, r') = 0 \), for \( r' \) in volume \( V \), we will choose \( F(r, r') \) to simplify solution of \( \phi \)

\[ \Rightarrow \frac{1}{V} \int d^3r' \left( \phi(r') \nabla'^2 G(r, r') - G(r', r') \nabla'^2 \phi(r') \right) \]

\[ = \frac{1}{V} \int d^3r' \left( \phi(r') \left[-4\pi \delta(r-r')\right] - G(r, r') \left[-4\pi \delta(r-r')\right] \right) \]

\[ = -4\pi \phi(r) + 4\pi \oint_{\partial S'} G(r, r') \rho(r') \]

\[ = \oint_{\partial S'} \left( \phi \frac{\nabla G}{\partial m'} - G \frac{\partial \phi}{\partial m'} \right) \]
\[ \phi (\vec{r}) = \int_V d^3 r' \, G (\vec{r}, \vec{r}') \, \phi (\vec{r}') + \oint_S d\alpha' \left( G (\vec{r}, \vec{r}') \frac{\partial \phi (\vec{r}')}{\partial \vec{m}'} - \phi (\vec{r}') \frac{\partial G (\vec{r}, \vec{r}')}{\partial \vec{m}'} \right) \]

Consider the Dirichlet boundary problem. If we can choose \( F (\vec{r}, \vec{r}') \) such that \( G (\vec{r}, \vec{r}') = 0 \) for \( \vec{r}' \) on the boundary surface \( S \), then the above simplifies to

\[ \left[ \phi (\vec{r}) = \int_V d^3 r' \, G_d (\vec{r}, \vec{r}') \, \phi (\vec{r}') - \oint_S d\alpha' \, \frac{\phi (\vec{r}')}{S} \frac{\partial G (\vec{r}, \vec{r}')}{\partial \vec{m}'} \right] \]

Since \( \phi (\vec{r}) \) is specified in \( V \), and \( \phi (\vec{r}) \) is specified on \( S \), above then gives desired solution for \( \phi (\vec{r}) \) inside volume \( V \).

Finally, \( G_d \) is therefore equivalent to finding an \( F (\vec{r}, \vec{r}') \) such that \( \nabla'^2 F (\vec{r}, \vec{r}') = 0 \) for \( \vec{r}' \) in \( V \) (solves Laplace eqn) and

\[ F (\vec{r}, \vec{r}') = \frac{-1}{|\vec{r} - \vec{r}'|} \text{ for } \vec{r}' \text{ on boundary surface } S' \]

Always exists unique solution for \( F \)
Next consider Neumann boundary problem.

One might think to find \( F(r, \vec{r}') \) such that \( \frac{\partial G(r, \vec{r}')}{\partial n'} = 0 \) on boundary surface. But this is not possible.

Consider \( \int V \Delta G(r, \vec{r}') \, d^3r' = \int V \cdot \vec{V}' \, G(r, \vec{r}') \, d^3r' \)

\[
= \int_S \vec{V}' \, G(r, \vec{r}') \cdot \hat{n} \, da'
\]

\[
= \int_S \frac{\partial G(r, \vec{r}')}{\partial n'} \, da' = -4\pi \text{ since } \Delta G = -4\pi \delta(r - r')
\]

So we can't have \( \frac{\partial G}{\partial n'} = 0 \) for \( r' \) on \( S' \)

Simplest choice is then \( \frac{\partial G(x, \vec{r}')}{\partial n'} = -4\pi \) for \( r' \) on \( S \)

Then

\[
\phi(r) = \int_V d^3r' \, G(r, \vec{r}') \, F(r') + \int_S \frac{da'}{4\pi} \, G(r, \vec{r}') \, \frac{\partial \phi(r)}{\partial n'}
\]

\[
+ \int_S \frac{da'}{4\pi} \, \phi(r') \left( \frac{-4\pi}{S'} \right)
\]

\[
\phi(r') = \int_V d^3r' \, G(r, \vec{r}') \, F(r') + \int_S \frac{da'}{4\pi} \, G(r, \vec{r}') \, \frac{\partial \phi(r)}{\partial n'}
\]

\[
+ \left< \phi \right>_S
\]

Since \( F(r) \) is specified in \( V \)

and \( \frac{\partial \phi}{\partial n} \) is specified on \( S' \)

constant = average value of \( \phi \) on surface \( S' \).

above gives solution \( \phi(r) \) in \( V \) within additive constant \( \left< \phi \right>_S \)

Since \( F = -\nabla \phi \) the constant \( \left< \phi \right>_S \) is of no consequence.
Finding $G_{N}(\bar{r}, \bar{r}')$ is therefore equivalent to finding
an $\bar{F}(\bar{r}, \bar{r}')$ such that

$$\nabla^2 \bar{F}(\bar{r}, \bar{r}') = 0 \text{ for } \bar{r}' \text{ in } V$$

and

$$\frac{\partial \bar{F}(\bar{r}, \bar{r}')}{\partial n'} = -\frac{4\pi}{S} \text{ for } \bar{r}' \text{ on surface } S'$$

always exists a unique solution (within additive constant)

While $G_{D}$ and $G_{N}$ always exist in principle, they
depend in detail on the shape of the surface $S'$ and
are difficult to find except for simple geometries

In proceeding we defined $G$ by

$$\nabla^2 \bar{G}(\bar{r}, \bar{r}') = -\frac{4\pi}{S} \delta(\bar{r}-\bar{r}')$$

But our earlier interpretation of $G(\bar{r}, \bar{r}')$ was that
it was potential at $\bar{r}$ due to point source at $\bar{r}'$, i.e.

$$\nabla^2 G(\bar{r}, \bar{r}') = -4\pi \delta(\bar{r}-\bar{r}')$$

Note, for general surface $S'$, $G(\bar{r}, \bar{r}')$ is not in general a function of $|\bar{r}-\bar{r}'|$ but
depends on $\bar{r}$ and $\bar{r}'$ separately. But the equivalence
of the two definitions of $G$ above is obtained by
noting that one can prove the symmetry property

$$G(\bar{r}, \bar{r}') = G(\bar{r}', \bar{r})$$

for Dirichlet b.c., and one can impose it as
an additional requirement for Neumann b.c.
(see Jackson, end section 1.10)