**Image Charge Method**

For simple geometries, can try to obtain \( G_D \) or \( G_W \) by placing a set of "image charges" outside the volume of interest \( V \), i.e. on the "other side" of the system boundary surface \( S \). Because these image charges are outside \( V \), their contribution to the potential inside \( V \) obeys \( \nabla^2 \phi_{\text{image}} = 0 \), as necessary. Choose location of image charges so that total \( \phi \) has desired boundary condition.

1) Charge in front of infinite grounded plane

\[
\begin{align*}
\phi (r) &= \frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{-q}{\sqrt{x^2 + y^2 + (z+d)^2}} \\
&= \frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{-q}{\sqrt{x^2 + y^2 + (z+d)^2}}
\end{align*}
\]

**Solution** - put fictitious image charge \(-q\) at \( z = -d \)

\( \phi \) is Coulomb potential from the real charge \(+q\) and the image charge \(-q\).

Above satisfies \( \phi (x, y, 0) = 0 \) as required.

**Also**,

\[
\nabla^2 \phi = -4\pi q \delta(r - d\hat{z}) + 4\pi q \delta (r + d\hat{z}) = -4\pi q \delta(r - d\hat{z})
\]

for \( r > 0 \).
Can now find \( E \) for \( z > 0 \)

\[
\dot{E} = -\frac{\partial \phi}{\partial \imath}
\]

In particular, \( E_z = -\frac{\partial \phi}{\partial z} = 8 \int \left( \frac{1}{2} \frac{2(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{2(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}} \right) \)

\[
E_z = \frac{8}{2} \left[ \frac{(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}} \right]
\]

We can use above to compute the surface charge density \( \sigma(x,y) \) induced on the surface of the conducting plane. At conductor surface

\[
-\frac{\partial \phi}{\partial n} = 4\pi \sigma
\]

\[
\Rightarrow \sigma = -\frac{1}{4\pi} \frac{\partial \phi}{\partial z} = \frac{1}{4\pi} E_z (x,y, z=0)
\]

\[
\sigma(x,y) = \frac{8}{4\pi} \left[ -\frac{d}{(x^2+y^2+d^2)^{3/2}} - \frac{d}{(x^2+y^2+d^2)^{3/2}} \right]
\]

\[
= -\frac{8d}{2\pi} \frac{1}{(x^2+y^2+d^2)^{3/2}} = -\frac{8d}{2\pi} \frac{1}{(r^2+d^2)^{3/2}}
\]

\[
\sigma = \frac{8d}{2\pi} \frac{1}{(x^2+y^2+d^2)^{3/2}}
\]

\[
\sigma = \frac{8d}{2\pi} \frac{1}{(r_1^2+d^2)^{3/2}}
\]

\[
\gamma = \sqrt{x^2+y^2}
\]
Total induced charge is

\[ q_{\text{induced}} = \iint_{-\infty}^{\infty} dxdy \sigma(x,y) \]

\[ = \int_{0}^{\infty} d\rho_{1} \rho_{1} (-qd) \frac{1}{2\pi \left( \rho_{1}^{2} + d^{2} \right)^{3/2}} \]

\[ = -\frac{qd}{d} \int_{0}^{\infty} \frac{-1}{\left( \rho_{1}^{2} + d^{2} \right)^{1/2}} d\rho_{1} \]

\[ = -\frac{qd}{d} \left[ 0 - \frac{-1}{d} \right] \]

\[ q_{\text{induced}} = -\frac{q}{d} \quad \text{induced charge = m"age charge} \]

Force on charge \( q \) in front of conducting plane is due to the induced \( \sigma \). The \( E \) fields of this \( \sigma \) is, for \( z > 0 \), the same as the \( E \) field of the image charge.

\[ \Rightarrow \vec{F} = -\frac{q^{2}}{(2d)^{2}} \hat{z} = -\frac{q^{2}}{4d^{2}} \hat{z} \quad \text{attractive} \]

Work done to move \( q \) into position from infinity is

\[ W = -\int_{\infty}^{d} \vec{F} \cdot d\vec{x} = -\int_{\infty}^{d} F_{z} dz \]
\[ W = \int_0^\infty \frac{d\zeta}{\zeta} \left( -\frac{\zeta^2}{4\zeta^2} \right) = -\frac{\zeta^2}{4\zeta} = -\frac{\zeta}{4} \]

\[ W < 0 \Rightarrow \text{energy released} \]

**Note:** The above is **not** the electrostatic energy that would be present if the image charge were real, i.e. it is **not** 
\[ \phi_{\text{image}}(r=\frac{d}{2}) = -\frac{q^2}{2d} \]

One way to see why is to note that as \( q \) is moved quasi-statically in towards the conductor plane, the image charge also must be moving to stay equidistant on the opposite side.
2) Point charge in front of a grounded (φ=0) conducting sphere.

Charge q placed a distance s from center of grounded conducting sphere of radius R.

Place image charge q' inside sphere so that the combined φ from q and q' vanishes on surface of sphere.

By symmetry, q' should lie on the same radial line as q does, call the distance of q' from the origin "a".

\[
\phi(\vec{r}) = \frac{q}{|\vec{r} - s \hat{\imath}|} + \frac{q'}{|\vec{r} - a \hat{\imath}|}
\]

\[
= \frac{q}{(r^2 + s^2 - 2rs \cos \Theta)^{1/2}} + \frac{q'}{(r^2 + a^2 - 2ra \cos \Theta)^{1/2}}
\]

Can we choose q' and a so that \(\phi(\vec{r}, \Theta) = 0\) for all \(\Theta\)?
\[
\phi(r, \theta) = \frac{q}{(r^2 + s^2 - 2sr \cos \theta)^{1/2}} + \frac{q}{(r^2 + a^2 - 2ar \cos \theta)^{1/2}}
\]

Make denominators look alike

\[r^2 + a^2 - 2ar \cos \theta = \frac{a}{s} \left( \frac{5}{a} r^2 + sa - 2sr \cos \theta \right)\]

Choose \(sa = R^2\), i.e. \(a = R^2/s\), then \(sr^2 = s^2\)

and then the denominator of the 2nd term is

\[
\left[ \frac{R^2}{s^2} (s^2 + r^2 - 2sr \cos \theta) \right]^{1/2} = \frac{r}{s} \left[ 3^2 + R^2 - 2sr \cos \theta \right]^{1/2}
\]

\[
\Rightarrow \phi(r, \theta) = \frac{q}{(r^2 + s^2 - 2sr \cos \theta)^{1/2}} + \frac{q(R/r)}{(r^2 + s^2 - 2sr \cos \theta)^{1/2}}
\]

So choose \(q'(R/r) = -q\) \(\Rightarrow q' = -q R/s\)

to get \(\phi(r, \theta) = 0\).

Solution is

\[
\phi(r, \theta) = \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} - \frac{q R/s}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}}
\]

\[
= \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} - \frac{q}{(s^2 r^2 + R^2 - 2rs \cos \theta)^{1/2}}
\]

Can get induced surface charge on sphere by

\[
4\pi \sigma = \mathbf{E} \cdot \mathbf{n} \bigg|_{r=R} = -\frac{\partial \phi}{\partial r} \bigg|_{r=R}
\]

see Jackson Eq (2.5) for result.
$$\sigma(\theta) = -\frac{q^2}{4\pi RS} \frac{1 - (R/s)^2}{(1 + (R/s)^2 - 2(R/s)\cos\theta)^{3/2}}$$

$$\sigma(\theta)$$ is greatest at $$\theta = 0$$, as one should expect.

Can integrate $$\sigma(\theta)$$ to get total induced charge. One finds

$$2\pi \int_0^\pi \delta \sin \theta R^2 \sigma(\theta) = q' = -q\frac{R}{s}$$

In general, total induced charge is sum of all image charges.

Force of attraction of charge to sphere

Force on $$q$$ is due to electric field from induced charge $$\sigma$$ which is the same as the electric field from the image charge $$q'$$.  

$$\vec{F} = \frac{q'\hat{r}}{(s-a)^2} = -\frac{q^2(R/s)\hat{r}}{(s-R^2/s)^2} = -\frac{q^2R s}{(s^2-R^2)^2}\hat{r}$$

Close to the surface of the sphere, $$s \approx R$$, so write $$s = R + d$$ where $$d \ll R$$. Then

$$\vec{F} = -\frac{q^2R s}{(s-R)^2(s+R)^2} \approx -\frac{q^2R(R+d)}{d^2(2R+d)^2} \approx -\frac{q^2}{4d^2}$$

get same result as for infinite flat grounded plane. When $$q$$ is so close to surface that $$d \ll R$$, the charge does not "see" the curvature of the surface.
for from the surface, \( s \gg R \)

\[
\hat{F} = \frac{q q' z^2}{(s-a)^2} = -\frac{q^2 RS}{(s^2 - R^2)^2} \hat{z} = -\frac{q^2 R}{s^3} \hat{z}
\]

\( F \sim \frac{1}{s^3} \) very different from flat plane
also different from point charge

Note: In preceding two problems, what we found was a \( \phi \) such that \( \nabla^2 \phi = -4\pi \delta(\vec{r} - \vec{r}_0) \), for a charge at \( \vec{r}_0 \)
and \( \phi = 0 \) on the boundary. Such a \( \phi \) is nothing more than \( G_0 \), the corresponding Green function for
Dirichlet boundary conditions.

Suppose now that instead of a grounded sphere we have a sphere with fixed net charge \( Q \).

We want to add new image charge to represent this case.
If we put \( q' = -\frac{Q}{R^2} \) at \( a = \frac{R}{s} \) as before, the boundary condition of \( \phi = \text{const} \) on surface \( \vec{r} = R \) is met, but the net charge on the sphere is \( q' \) (the induced charge) not the desired \( Q \). We therefore need to add new image charges \( s \) of total charge \( Q - q' \) (so total image charge \( sQ \)) in such a way that we keep \( \phi \) constant on the surface of the sphere. The way to do this is to put \( Q - q' \) at the origin.
Solution 1:

$$\phi(r, \theta) = \frac{q + q R/s}{r} \left[ \frac{1}{s^3} - \frac{1}{s^3(1 - R^2/s^2)^2} \right]$$

The force on the charge $q$ is due to the $\vec{E}$ field of the images.

$$\vec{F} = \vec{F}^z = \frac{q (q + q R/s)^2}{s^2} + \frac{q q}/(s-a)^2$$

$$F = \frac{q A}{s^2} + \frac{q^2 R/s}{s^2} - \frac{q^2 R/s}{(s - R^2/s^2)^2}$$

$$= \frac{q A}{s^2} + \frac{q^2 R}{s^3} \left[ 1 - \frac{1}{(1 - R^2/s^2)^2} \right]$$

$$= \frac{q A}{s^2} + \frac{q^2 R}{s^3} \left[ 1 - \frac{1}{(1 - R^2/s^2)^2} \right]$$

$$F = \frac{q A}{s^2} - \frac{q^2 R^3}{s} \frac{2 - R^2/s^2}{(s^2 - R^2)^2}$$

For large $s \gg R$ for from surface

$$F \approx \frac{q A}{s^2} - \frac{2 q^2 R^3}{s^5}$$

leading term is just

Coulomb force between $q$

and $A$ at origin

for $A > 0$, $F$ is always repulsive for large enough $s$. 
For $s = R + d$, $d \ll R$ close to surface

$$F = \frac{qA}{(R + d)^2} - \frac{q^2 R^3}{(R + d)} z \left( \frac{R^2}{(R + d)^2} \right) \frac{2 - \frac{R^2}{(R + d)^2}}{(R^2 + d^2 + 2Rd - R^2)^2}$$

$$\approx \frac{qA}{R^2} - \frac{q^2 R^3}{R} \frac{(2 - 1)}{4R^2 d^2}$$

$$F \approx \frac{qA}{R^2} - \frac{q^2}{4d^2} \approx -\frac{q^2}{4d^2} \text{ for } d \text{ small enough}$$

$F$ is always attractive for small enough $d$, and is equal to the force in front of a grounded plane, no matter what is the value of $Q$. This is because the image charge $Q'$ lies so much close to $q$ than does the $Q-q$ at the origin, that it dominates the force.

The cross over from attractive to repellent occurs at a distance $s$ that depends on $Q$. This distance is given by

$$\frac{Q}{\varphi} = \frac{R^3 s (2 - R^2/s^2)}{(s^2 - R^2)^2} \left( \frac{R^3}{s} \right) \frac{z - (R/s)^2}{[1 - (R/s)^2]^2}$$

Let $x = R/s \in (0,1)$

$$\frac{Q}{\varphi} = x^3 \left(2 - x^2\right) \left(1 - x^2\right)^2$$

gives 5th order polynomial in $x$

no analytic solution

can solve graphically
For $\frac{Q}{S} = 1$, crossover is at $\frac{R}{S} = 0.62$

$S = 1.6R$

$\frac{Q}{S} = 0.1$, crossover is at $\frac{R}{S} = 0.36$

$S = 2.8R$