Spherical Coordinates

\[ \nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \phi^2} = 0 \]

\[ \phi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi) \]

\[ r^2 \nabla^2 \phi = \Theta \Phi \frac{1}{r \Theta} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{R \Phi}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \frac{R \Theta}{\sin^2 \theta} \frac{d^2 \Phi}{d\phi^2} = 0 \]

\[ \frac{r^2 \sin^2 \theta \nabla^2 \phi}{\phi} = \frac{\sin^2 \theta}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0 \]

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depends only on \( r \) and \( \theta \) = \( -\text{const} \)

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depends only on \( \phi \) = \( \text{const} \)

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Take \( \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = -m^2 \)

\[ \Rightarrow \Phi = e^{\pm i m \phi} \]

\( m \) integer for 2\pi periodicity with \( \phi \)

\[ \Rightarrow \frac{\sin^2 \theta}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = m^2 \]

\[ \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} = 0 \]

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depends only on \( r \)

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depends only on \( \theta \)

\[ = \text{const} \]

\[ = -\text{const} \]
Call the constant \( L(l+1) \)

For \( \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = L(l+1) \):

Solutions are of the form \( R(r) = a_0 \, r^l + b_0 \, r^{-(l+1)} \)

Substitute \( R \) to verify

\[
\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \frac{d}{dr} \left( r^2 \left( la_0 \, r^{l-1} - (l+1) b_0 \, r^{-l-2} \right) \right) = \frac{d}{dr} \left( la_0 \, r^{l+1} - (l+1) b_0 \, r^{-l} \right) = l(l+1) a_0 \, r^l + l(l+1) b_0 \, r^{-l-1} = l(l+1) R
\]

For \( \Theta \):

\[
\frac{1}{\sin \Theta} \frac{d}{d\Theta} \left( \sin \Theta \frac{d\Theta}{d\Theta} \right) = \frac{m^2}{\sin^2 \Theta} = -l(l+1)
\]

Let \( x = \cos \Theta \)

\[
dx = -\sin \Theta \, d\Theta
\]

\[
\frac{d\Theta}{\sin \Theta} = -\frac{dx}{\sin \Theta}
\]

Solutions for \(-1 \leq x \leq 1\) correspond to \( l \geq 0 \) integers

\[
\frac{d}{dx} \left[ (1-x^2) \frac{d\Theta}{dx} \right] + \left[ l(l+1) - \frac{m^2}{1-x^2} \right] \Theta = 0
\]

Called generalized Legendre Equation - Solutions are called the associated Legendre functions.

Ordinary Legendre polynomials are solutions for \( m = 0 \)
For the special case \( m = 0 \), i.e. the solution has azimuthal symmetry and \( \Phi \) does not depend on the angle \( \phi \) (i.e. rotational symmetry about z-axis),

we want the solutions to

\[
\frac{d}{dx} \left[ (1-x^2) \frac{d\Theta}{dx} \right] + \ell (\ell + 1) \Theta = 0
\]

The solutions are known as the Legendre polynomials, \( P_\ell(x) \).

They are given, for \( \ell \) integer, by

\[
P_\ell(x) = \frac{1}{2^\ell \ell!} \left( \frac{d}{dx} \right)^\ell (x^2 - 1)^\ell \quad \text{Rodriguez's formula}
\]

The lowest \( \ell \) polynomials are

\[
P_0(x) = 1, \quad P_2(x) = \frac{1}{2} (3x^2 - 1),
\]

\[
P_1(x) = x, \quad P_3(x) = \frac{1}{2} (5x^3 - 3x)
\]

In general, \( P_\ell(x) \) is a polynomial of order \( \ell \) with only even powers of \( x \) if \( \ell \) is even, and only odd powers of \( x \) if \( \ell \) is odd. \( \Rightarrow P_\ell(x) \) is even in \( x \) for \( \ell \) even and odd in \( x \) for \( \ell \) odd.

\( P_\ell(x) \) is normalized so that \( P_\ell(1) = 1 \).
Legendre polynomials are only for integer \( l \geq 0 \). What about solutions for non-integer \( l \)?

The \( P_l(x) \) give one solution for each integer \( l \).

But \( P_l(x) \) are defined by a 2nd order differential equation - shouldn't there be a 2nd independent solution for each \( l \)?

It turns out that these "2nd" solutions, as well as solutions for non-integer \( l \), all blow up at either \( x = -1 \) or \( x = 1 \), i.e. at \( \theta = 0 \) or \( \theta = \pi \).

They therefore are physically unacceptable and we do not need to consider them. See Jackson 3.2.

The Legendre polynomials are orthogonal and form a complete set of basis functions on the interval \(-1 \leq x \leq 1\).

\[
\int_{-1}^{1} P_l(x) P_m(x) \, dx = \int_0^\pi \sin \theta \, P_l(\cos \theta) P_m(\cos \theta) \, d\theta = \begin{cases} 0 & l \neq m \\frac{2}{2l+1} & l = m \end{cases}
\]

\[
\Rightarrow \text{we can expand any function } f(\theta), \ 0 < \theta < \pi,
\]

as a linear combination of the \( P_l(\cos \theta) \).

This is the reason they are useful for solving problems of Laplace's type with spherical boundary surfaces.
For \( m \neq 0 \), the solutions to
\[
\frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} \right] + \left[ \ell(\ell+1) - \frac{m^2}{1-x^2} \right] \Theta = 0
\]
are the associated Legendre functions \( P_\ell^m(x) \).

For \( P_\ell^m(x) \) to be finite in interval \(-1 \leq x \leq 1\)
one again finds that \( \ell \) must be integer \( \ell \geq 0 \), and integer \( m \) must satisfy \( |m| \leq \ell \), i.e., \( m = -\ell, -\ell+1, \ldots, 0, \ldots, \ell-1, \ell \).

For each \( \ell \) and \( m \) there is only one such non-divergent solution.

It is typical to combine the solutions \( P_\ell^m(\cos \theta) \) to the
\( \theta \)-part of the equation with the \( \Phi_\ell^m(\phi) = e^{i m \phi} \) solutions
to the \( \phi \)-part of the equation to define the spherical harmonics
\[
Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\phi}
\]
The \( Y_{\ell m} \) are orthogonal
\[
\int_0^{2\pi} \int_0^\pi \sin \theta \ d\theta \ d\phi \ Y_{\ell' m'}^*(\theta, \phi) Y_{\ell m}(\theta, \phi) = \delta_{\ell \ell'} \delta_{m m'}
\]
and are a complete set of basis functions for expanding any function \( f(\theta, \phi) \) defined on the surface of a sphere.
Behavior of fields near a central hole or sharp tip

We now want to solve the $\nabla^2 \phi = 0$ with separation of variables, but now $\theta$ is restricted to range $0 \leq \theta \leq \beta$.

We still have azimuthal symmetry, but now, since we do not need solution to $\theta$ be finite for all $0 \leq \theta \leq \pi$, but only $\theta \in (0, \beta)$, we have more solutions to the $\theta$ equation, i.e. it does not have to be integer. Still need $d > 0$ to be finite at $\theta = 0$.

See Jackson sec. 3.4 for details.
Examples with azimuthal symmetry \( m = 0 \)

General solution to \( \nabla^2 \phi = 0 \) can be written in form

\[
\phi(r, \theta) = \sum_{\ell=0}^{\infty} \left[ A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right] P_\ell (\cos \theta)
\]

determine the \( A_\ell \) and \( B_\ell \) from the boundary conditions of the particular problem.

1. Suppose one is given \( \phi(R, \theta) = \phi_0 (\theta) \) on surface of sphere of radius \( R \).

To find solution of \( \nabla^2 \phi = 0 \) inside sphere,

\( \phi \) should not diverge at origin \( \Rightarrow B_\ell = 0 \) for all \( \ell \)

\[
\phi(r, \theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell (\cos \theta)
\]

\( \Rightarrow \phi(r, \theta) = \phi_0 (r) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell (\cos \theta) \)

\[
\int_0^\pi \cos \theta \phi_0 (\theta) P_m (\cos \theta) \, d\theta = \sum_{\ell=0}^{\infty} A_\ell R^\ell \int_0^\pi \cos \theta P_\ell (\cos \theta) P_m (\cos \theta) \, d\theta
\]

\[
= \sum_{\ell=0}^{\infty} A_\ell R^\ell \left( \frac{2}{2\ell+1} \right) \delta_{\ell m}
\]

\[
A_m = \frac{2m+1}{2R^m} \int_0^\pi \cos \theta \phi_0 (\theta) P_m (\cos \theta) \, d\theta
\]

swiss solution
To find solution of $\nabla^2 \phi = 0$ outside sphere

If require $\phi \to 0$ as $r \to 0$, then $A_\ell = 0$ for all $\ell$

$$\phi(r, \theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos \theta)$$

$$\phi(R, \theta) = \phi_0(\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos \theta)$$

Gives solution

$$B_\ell = \frac{2m+1}{2} R^{m+1} \int_0^\pi \sin \theta \phi_0(\theta) P_m(\cos \theta)$$

$$B_\ell = A_m R^{2m+1}$$

(2) Suppose one is given surface charge density $\sigma(\theta)$ fixed on surface of sphere of radius $R$. What is $\phi$ inside and outside?

From previous example

$$\phi(r, \theta) = \begin{cases} \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos \theta) & r < R \\ \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos \theta) & r > R \end{cases}$$

Boundary conditions at $r = R$ on surface

(i) $\phi$ continuous

$$\sum_{\ell=0}^{\infty} \left[ A_\ell R^\ell - \frac{B_\ell}{R^{\ell+1}} \right] P_\ell(\cos \theta) = 0$$
If an expansion in Legendre polynomials vanishes for all $\theta$, then each coefficient in the expansion must vanish.

\[ A e R^{-l} = \frac{B_l}{R^{l+1}} \Rightarrow B_l = A e R^{2l+1} \]

(iii) Jump in electric field at $\sigma$

\[ -\frac{\partial \phi^\text{out}}{\partial r} \bigg|_{r = R} + \frac{\partial \phi^\text{in}}{\partial r} \bigg|_{r = R} = 4\pi \sigma \]

\[ \Rightarrow \sum_{l=0}^{\infty} \frac{(l+1) B_l}{R^{l+2}} + l A e R^{l-1} \int p_e(\cos \theta) = 4\pi \sigma \]

\[ \Rightarrow \sum_{l=0}^{\infty} \left[ \frac{(l+1) A e R^{2l+1}}{R^{l+2}} + l A e R^{l-1} \right] p_e(\cos \theta) \]

\[ \Rightarrow \sum_{l=0}^{\infty} (2l+1) R^{2l+1} A e \int p_e(\cos \theta) = 4\pi \sigma \]

\[ (2m+1) R^{m-1} A_m \left( \frac{2}{2m+1} \right) = 4\pi \int_0^\pi d\theta \sin \theta \sigma(\theta) P_m(\cos \theta) \]

\[ A_m = \frac{4\pi}{2 R^{m-1}} \int_0^\pi d\theta \sin \theta \sigma(\theta) P_m(\cos \theta) \]
Suppose $\sigma(\theta) = k \cos \theta$. What is $\phi$?

Note $\sigma(\theta) = k P_1(\cos \theta)$

hence only $A_1 \neq 0$ by orthogonality of $P_n(\cos \theta)$

$$A_1 = \frac{4\pi k}{2} \int_0^\pi \sin \theta \, P_1(\cos \theta) \, P_1(\cos \theta) \, d\theta$$

$$= \frac{4\pi k}{2} \left( \frac{2}{2+1} \right) = \frac{4\pi k}{3}$$

$\Rightarrow \phi(r, \theta) = \begin{cases} 
\frac{4\pi k}{3} kr \cos \theta & r < R \\
\frac{4\pi k}{3} \frac{R^3}{r^2} \cos \theta & r > R 
\end{cases}$

We will see that potential outside the sphere is that of an ideal dipole with dipole moment $p = \frac{4\pi R^3 k}{3}$.

Inside the sphere, the potential $\phi = \frac{4\pi k}{3} \frac{z}{r}$, where $z = r \cos \theta$. The electric field inside the sphere is therefore the constant $E = -\nabla \phi = -\frac{4\pi k}{3} \frac{z}{r}$.
outside the sphere the field is

\[ \vec{E} = -\vec{\nabla}\phi = -\frac{\partial \phi}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} \]

\[ = \frac{8\pi k R^3}{r^3} \cos \theta \hat{r} + \frac{4\pi k R^3}{r^3} \sin \theta \hat{\theta} \]

\[ \vec{E} = \frac{4\pi k R^3}{r^3} \left( \cos \theta \hat{r} + \sin \theta \hat{\theta} \right) \]

\[ \]
Physical example with $\sigma(\theta) = \kappa \cos \theta$

Two spheres of radii $R$, with equal but opposite uniform charge densities $\rho$ and $-\rho$, displaced by small distance $d \ll R$

Surface charge $\sigma$ builds up due to displacement. This is a uniformly "polarized" sphere.

Surface charge is $\sigma(\theta) = \rho \, Sr$

$\sigma(\theta) = \rho \, d \, \cos \theta$

Total dipole moment is $(\rho d) \frac{4\pi R^3}{3}$

Polarization = dipole moment \(\frac{\text{volume}}{\text{volume}} = \rho d$

$\vec{E}$ field inside a uniformly polarized sphere is constant: $\vec{E} = \rho d \frac{4\pi}{3}$