Spherical Coordinates

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0$$

$$\phi(r,\theta,\varphi) = R(r)\,\Theta(\theta)\,\Phi(\varphi)$$

$$r^2\nabla^2\phi = \Theta\Phi\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{R\Theta}{\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = 0$$

$$\frac{r^2\sin^2\theta\,\nabla^2\phi}{\phi} = \underbrace{\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{\substack{\text{depends only on } r \text{ and } \theta \\ = -\,\text{const}}} + \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}_{\substack{\text{depends only on } \varphi \\ = \text{const}}} = 0$$

take $\displaystyle \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -m^2$

$$\Rightarrow \boxed{\Phi = e^{\pm i m\varphi}}$$

m integer for 2π periodicity in φ

$$\Rightarrow \frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2$$

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\substack{\text{depends only on } r \\ = \text{const}}} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}}_{\substack{\text{depends only on } \theta \\ = -\,\text{const}}} = 0$$

call the const $l(l+1)$

For R

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - l(l+1) = 0$$

Solutions are of the form $\boxed{R(r) = a_l r^l + b_l r^{-(l+1)}}$

substitute in to verify

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{d}{dr}\left(r^2\left(l a_l r^{l-1} - (l+1) b_l r^{-l-2}\right)\right)$$

$$= \frac{d}{dr}\left(l a_l r^{l+1} - (l+1) b_l r^{-l}\right)$$

$$= l(l+1) a_l r^l + l(l+1) b_l r^{-(l+1)} = l(l+1) R$$

For Θ:

$$\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -l(l+1)$$

let $x = \cos\theta$

$dx = -\sin\theta \, d\theta$

$d\theta = -\dfrac{dx}{\sin\theta}$

$0 \leq \theta \leq \pi$

solutions for $-1 \leq x \leq 1$ correspond to $l \geq 0$ integers

above becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]\Theta = 0$$

Called generalized Legendre Equation - solutions are called the associated Legendre functions.

ordinary Legendre polynomials are solutions for $m=0$

For the special case $m=0$, ie the solution has azimuthal symmetry and Φ does not depend on the angle φ (ie rotational symmetry about $\hat{z}$ axis),

We want the solutions to

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \ell(\ell+1)\,\Theta = 0$$

The solutions are known as the Legendre polynomials, $P_\ell(x)$.

They are given, for ℓ integer, by

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell \qquad \text{Rodriguez's formula}$$

The lowest ℓ polynomials are

$$P_0(x) = 1 \qquad\qquad P_2(x) = \tfrac{1}{2}(3x^2-1)$$
$$P_1(x) = x \qquad\qquad P_3(x) = \tfrac{1}{2}(5x^3-3x)$$

In general, $P_\ell(x)$ is a polynomial of order ℓ with only even powers if ℓ is even, and only odd powers if ℓ is odd. $\Rightarrow P_\ell(x) \begin{cases} \text{even in } x \text{ for } \ell \text{ even} \\ \text{odd in } x \text{ for } \ell \text{ odd} \end{cases}$

$P_\ell(x)$ is normalized so that $P_\ell(1) = 1$

Note: Legendre polynomials are only for integer $\ell \geq 0$.
What about solutions for non integer ℓ?
The $P_\ell(x)$ give one solution for each integer ℓ.
But $P_\ell(x)$ are defined by a 2nd order differential equation - shouldn't there be a 2nd independent solution for each ℓ?
It turns out that these "2nd" solutions, as well as solutions for non integer ℓ, all blow up at either $x=-1$ or $x=1$, ie at $\theta=0$ or $\theta=\pi$.
They therefore are physically unacceptable and we do not need to consider them. See Jackson 3.2

The Legendre polynomials are orthogonal and form a complete set of basis functions on the interval $-1 \leq x \leq 1$.

$$\int_{-1}^{1} dx\, P_\ell(x) P_m(x) = \int_0^\pi d\theta \sin\theta\, P_\ell(\cos\theta) P_m(\cos\theta) = \begin{cases} 0 & \ell \neq m \\ \dfrac{2}{2\ell+1} & \ell = m \end{cases}$$

$\Rightarrow$ we can expand any function $f(\theta)$, $0 \leq \theta \leq \pi$, as a linear combination of the $P_\ell(\cos\theta)$.
This is the reason they are useful for solving problems of Laplace's eqn with spherical boundary surfaces.

For $m \neq 0$, the solutions to (see Jackson 3.5)

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]\Theta = 0$$

are the associated Legendre functions $P_\ell^m(x)$.
For $P_\ell^m(x)$ to be finite in interval $-1 \leq x \leq 1$ one again finds that ℓ must be integer $\ell \geq 0$, and integer m must satisfy $|m| \leq \ell$, ie $m = -\ell, -\ell+1, \ldots, 0, \ldots \ell-1, \ell$.
For each ℓ and m there is only one such non divergent solution.

It is typical to combine the solutions $P_\ell^m(\cos\theta)$ to the θ-part of the equation with the $\Phi_m(\varphi) = e^{im\varphi}$ solutions to the φ-part of the equation to define the spherical harmonics

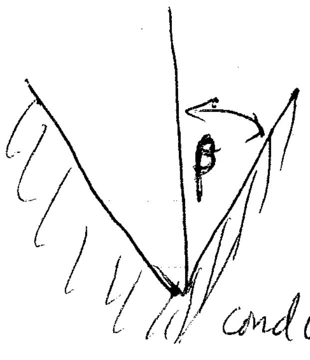
$$Y_{\ell m}(\theta, \varphi) \equiv \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\varphi}$$

The $Y_{\ell m}$ are orthogonal

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \; Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}$$

and are a complete set of basis functions for expanding any function $f(\theta,\varphi)$ defined on the surface of a sphere.

Behavior of fields near conical hole or sharp tip



we now want to solve the $\nabla^2\phi = 0$ with separation of variables, but now θ is restricted to range $0 \le \theta \le \beta$.

We still have azimuthal symmetry, but now, since we do not need solution to be finite for all $\theta \in [0, \pi]$, but only $\theta \in (0, \beta)$, we have more solutions to the Θ equation, ie l does not have to be integer. - still need $l \geq 0$ to be finite at $\theta = 0$.

see Jackson sec. 3.4 for details.

Exaples with azimuthal symmetry $m=0$

General solution to $\nabla^2\phi = 0$ can be written in form

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right] P_\ell(\cos\theta)$$

determine the A_ℓ and B_ℓ from the boundary conditions of the particular problem.

① Suppose one is given $\phi(R,\theta) = \phi_0(\theta)$ on surface of sphere of radius R.

To find solution of $\nabla^2\phi = 0$ inside sphere

ϕ should not diverge at origin $\Rightarrow$ $B_\ell = 0$ for all ℓ

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$\Rightarrow \phi(R,\theta) = \phi_0(R) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta)$$

$$\rightarrow \int_0^\pi d\theta \sin\theta \, \phi_0(\theta) P_m(\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell \int_0^\pi d\theta \sin\theta \, P_\ell(\cos\theta) P_m(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} A_\ell R^\ell \left(\frac{2}{2\ell+1}\right) \delta_{\ell m}$$

$$= A_m R^m \frac{2}{2m+1}$$

$$\boxed{A_m = \frac{2m+1}{2R^m} \int_0^\pi d\theta \sin\theta \, \phi_0(\theta) P_m(\cos\theta)}$$

gives solution

To find solution of $\nabla^2\phi = 0$ <u>outside</u> sphere

if require $\phi \to 0$ as $r \to 0$, then $A_\ell = 0$ for all ℓ

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$\phi(R,\theta) = \phi_0(\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

gives solution
$$\boxed{B_m = \frac{2m+1}{2} R^{m+1} \int_0^\pi d\theta \sin\theta\, \phi_0(\theta) P_m(\cos\theta)}$$

$$\boxed{B_m = A_m R^{2m+1}}$$

2 Suppose one is given surface charge density $\sigma(\theta)$ fixed on surface of sphere of radius R. What is ϕ inside and outside?

From previous example

$$\phi(r,\theta) = \begin{cases} \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) & r < R \\ \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) & r > R \end{cases}$$

boundary conditions at $r = R$ on surface

(i) ϕ continuous

$$\Rightarrow \sum_{\ell=0}^{\infty} \left[A_\ell R^\ell - \frac{B_\ell}{R^{\ell+1}} \right] P_\ell(\cos\theta) = 0$$

If an expansion in Legendre polynomials vanishes for all θ, then each coefficient in the expansion must vanish

$$\Rightarrow \quad A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}} \quad \Rightarrow \boxed{B_\ell = A_\ell R^{2\ell+1}}$$

(ii) jump in electric field at σ

$$-\frac{\partial \phi^{out}}{\partial r}\Big|_{r=R} + \frac{\partial \phi^{in}}{\partial r}\Big|_{r=R} = 4\pi\sigma$$

$$\Rightarrow \sum_{\ell=0}^{\infty}\left[\frac{(\ell+1)B_\ell}{R^{\ell+2}} + \ell A_\ell R^{\ell-1}\right]P_\ell(\cos\theta) = 4\pi\sigma$$

$$\Rightarrow \sum_{\ell=0}^{\infty}\left[\frac{(\ell+1)A_\ell R^{2\ell+1}}{R^{\ell+2}} + \ell A_\ell R^{\ell-1}\right]P_\ell(\cos\theta)$$

$$\Rightarrow \sum_{\ell=0}^{\infty}(2\ell+1)R^{\ell-1}A_\ell P_\ell(\cos\theta) = 4\pi\sigma$$

$$(2m+1)R^{m-1}A_m\left(\frac{2}{2m+1}\right) = 4\pi\int_0^\pi d\theta \sin\theta\, \sigma(\theta) P_m(\cos\theta)$$

$$\boxed{A_m = \frac{4\pi}{2R^{m-1}}\int_0^\pi d\theta \sin\theta\, \sigma(\theta)\, P_m(\cos\theta)}$$

Suppose $\sigma(\theta) = k\cos\theta$ what is ϕ?

Note $\sigma(\theta) = k P_1(\cos\theta)$

hence only $A_1 \neq 0$ by orthogonality of $P_\ell(\cos\theta)$

$$A_1 = \frac{4\pi k}{2} \int_0^\pi d\theta \sin\theta \, P_1(\cos\theta) P_1(\cos\theta)$$

$$= \frac{4\pi k}{2} \left(\frac{2}{2+1} \right) = \frac{4}{3}\pi k$$

$$\Rightarrow \phi(r,\theta) = \begin{cases} \frac{4\pi}{3} k \, r\cos\theta & r < R \\ \frac{4\pi}{3} k \frac{R^3}{r^2} \cos\theta & r > R \end{cases}$$

we will see that potential <u>outside</u> the sphere is that of an ideal dipole with dipole moment

$p = \frac{4}{3}\pi R^3 k$

Inside the sphere, the potential $\phi = \frac{4\pi}{3} k z$ where $z = r\cos\theta$. The electric field inside the sphere is therefore the constant

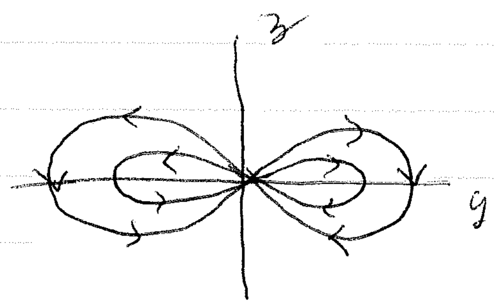
$$\vec{E} = -\vec{\nabla}\phi = -\frac{4}{3}\pi k \hat{z}$$

outside the sphere the field is

$$\vec{E} = -\vec{\nabla}\phi = -\frac{\partial\phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta}$$

$$= \frac{8\pi}{3} k \frac{R^3}{r^3}\cos\theta\,\hat{r} + \frac{4\pi}{3} k \frac{R^3}{r^3}\sin\theta\,\hat{\theta}$$

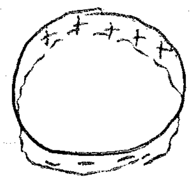
$$\vec{E} = \frac{4\pi}{3} R^3 k \frac{1}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$



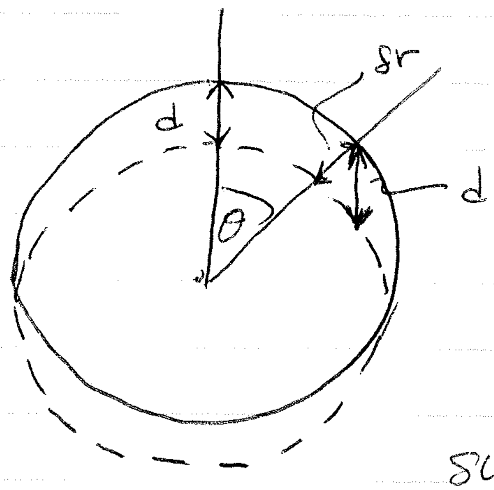
dipole field

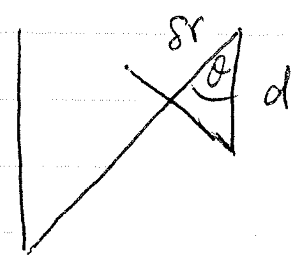
Physical exaple with $\sigma(\theta) = k\cos\theta$

Two spheres of radii R, with equal but opposite uniform charge densities ρ and $-\rho$, displaced by small distance $d \ll R$



surface charge σ builds up due to displacement. This is a uniformly "polarized" sphere





$$d\cos\theta = \delta r$$

surface charge is $\sigma(\theta) = \rho\,\delta r$

$$= \rho d \cos\theta$$

$$\boxed{\sigma(\theta) = \rho d \cos\theta}$$

total dipole moment is $(\rho d)\frac{4}{3}\pi R^3$

$$\text{polarization} = \frac{\text{dipole moment}}{\text{volume}} = \rho d$$

$\vec{E}$ field inside a uniformly polarized sphere is constant. $\vec{E} = -\rho d \frac{4\pi}{3}$