$$\vec{E} = \frac{1}{r^3}\left[\, 3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\,\right]$$

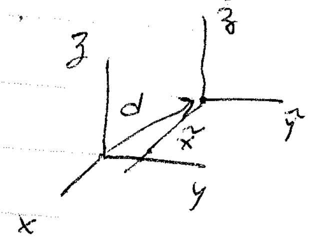
expresses $\vec{E}$ of dipole in coord free form

<u>Origin of coordinates</u>

The definition of the multipole moments <u>depends</u> on the choice of origin of the coordinates

Suppose transform to $\tilde{\vec{r}} = \vec{r} - \vec{d}$

In the $\tilde{\vec{r}}$ cood system



$$\tilde{q} = \int d^3\tilde{r}\ \rho(\tilde{r}) = \int d^3r\ \rho(r) = q$$

monopole <u>does not</u> depend on choice of origin

$$\tilde{\vec{p}} = \int d^3\tilde{r}\ \rho(\tilde{r})\ \tilde{\vec{r}} = \int d^3r\ \rho\ (\vec{r} - \vec{d})$$

$$= \int d^3r\ \rho\ \vec{r} - \vec{d}\int d^3r\ \rho$$

$$\tilde{\vec{p}} = \vec{p} - \vec{d}q \qquad \tilde{\vec{p}} = \vec{p} \text{ only if } q = 0\,!$$

if $q \neq 0$, then $\tilde{\vec{p}} \neq \vec{p}$

$\Rightarrow$ ~~One coord~~ If $q \neq 0$, one could always choose an origin of coords for which $\vec{p} = 0$!

~~For~~ HW you will show that $\tilde{\vec{p}}$ ...

Quadrupole moment in new coordinates

$$\overleftrightarrow{\tilde{Q}} = \int d^3\tilde{r}\, \rho \left[3\vec{\tilde{r}}\,\vec{\tilde{r}} - (\vec{\tilde{r}})^2 \overleftrightarrow{I} \right]$$

where $\vec{\tilde{r}} = \vec{r} - \vec{d}$

substitute in above

$$\overleftrightarrow{\tilde{Q}} = \int d^3r\, \rho \left[3(\vec{r}-\vec{d})(\vec{r}-\vec{d}) - (\vec{r}-\vec{d})^2 \overleftrightarrow{I} \right]$$

$$= \int d^3r\, \rho \left[3\vec{r}\,\vec{r} - 3\vec{r}\,\vec{d} - 3\vec{d}\,\vec{r} + 3\vec{d}\,\vec{d} - (r^2 + d^2 - 2\vec{r}\cdot\vec{d})\overleftrightarrow{I} \right]$$

$$= \int d^3r\, \rho \left[3\vec{r}\,\vec{r} - r^2 \overleftrightarrow{I} \right] - 3\left[\int d^3r\, \rho\, \vec{r}\right]\vec{d} - 3\vec{d}\left[\int d^3r\, \rho\, \vec{r}\right]$$

$$+ 3\vec{d}\,\vec{d}\left[\int d^3r\, \rho\right] - d^2 \overleftrightarrow{I}\left[\int d^3r\, \rho\right]$$

$$+ 2\left[\int d^3r\, \rho\, \vec{r}\right]\cdot\vec{d}\,\overleftrightarrow{I}$$

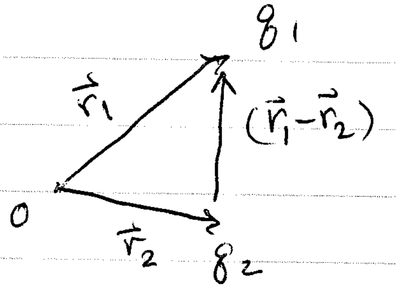
$$\overleftrightarrow{\tilde{Q}} = \overleftrightarrow{Q} - 3\vec{p}\,\vec{d} - 3\vec{d}\,\vec{p} + 3\vec{d}\,\vec{d}\,q - (d^2 q - 2\vec{p}\cdot\vec{d})\overleftrightarrow{I}$$

we see that $\overleftrightarrow{Q}$ is independent of choice of origin only when both q and $\vec{p}$ vanish. When this happens the quadrupole term is the <u>leading</u> term in the multipole expansion.

In general, the leading term in multipole expansion will be indep of origin of coordinates.

Example two charges q_1 at $\vec{r}_1$ and q_2 at $\vec{r}_2$

$$q_1 + q_2 = q \neq 0$$



monopole $q_1 + q_2 = q$

dipole $\vec{p} = q_1 \vec{r}_1 + q_2 \vec{r}_2$

quadrupole $\overleftrightarrow{Q} = (3\vec{r}_1\vec{r}_1 - r_1^2 \overleftrightarrow{I})q_1 + (3\vec{r}_2\vec{r}_2 - r_2^2 \overleftrightarrow{I})q_2$

We can make the dipole moment vanish by shifting to a new coord system $\vec{r}\,' = \vec{r} - \vec{d}$ where $\vec{d} = \dfrac{\vec{p}}{q}$

$$\vec{r}\,' = \vec{r} - \frac{q_1\vec{r}_1 + q_2\vec{r}_2}{q_1 + q_2} = \frac{q_1(\vec{r} - \vec{r}_1) + q_2(\vec{r} - \vec{r}_2)}{q_1 + q_2}$$

positions of q_1, q_2 in new coords are

$$\vec{r}_1\,' = \frac{q_2}{q_1 + q_2}(\vec{r}_1 - \vec{r}_2)$$

$$\vec{r}_2\,' = \frac{-q_1}{q_1 + q_2}(\vec{r}_1 - \vec{r}_2)$$

Origin of new coord system is at

lies along vector from $\vec{r}_2$ to $\vec{r}_1$

$$\vec{r}\,' = 0 \Rightarrow \vec{r} = \frac{q_1\vec{r}_1 + q_2\vec{r}_2}{q_1 + q_2}$$ "center of charge"

for many charges q_i at positions $\vec{r}_i$, the origin that makes dipole moment vanish is at

$$\vec{r} = \frac{\sum_i q_i \vec{r}_i}{\sum q_i}$$

In this coord system

$$\vec{p}\,' = q_1\vec{r}_1\,' + q_2\vec{r}_2\,' = \frac{q_1 q_2}{q_1+q_2}(\vec{r}_1-\vec{r}_2) - \frac{q_2 q_1}{q_1+q_2}(\vec{r}_1-\vec{r}_2)$$

$$= 0 \quad \text{as it must be!}$$

<u>Quadrupole moment</u> in the coord system in which $\vec{p}\,'=0$ the quadrupole tensor is

$$\overleftrightarrow{Q}' = \left[3\vec{r}_1\,'\vec{r}_1\,' - (r_1')^2\overleftrightarrow{I}\right]q_1 + \left[3\vec{r}_2\,'\vec{r}_2\,' - (r_2')^2\overleftrightarrow{I}\right]q_2$$

let us choose ~~coord~~ spherical coordinates with origin at O' and $\hat{z}$ axis aligned along $\vec{r}_1-\vec{r}_2$, so that

$$\vec{r}_1-\vec{r}_2 = s\,\hat{z} \quad \text{where } s = |\vec{r}_1-\vec{r}_2| \text{ is separation between the charges}$$

Then

$$\vec{r}_1\,' = \frac{q_2}{q_1+q_2}\,s\,\hat{z}$$

$$\vec{r}_2\,' = \frac{-q_1}{q_1+q_2}\,s\,\hat{z}$$

$$\overleftrightarrow{Q}' = \left(\frac{q_2}{q_1+q_2}\right)^2 q_1\left[3s^2\hat{z}\hat{z} - s^2\overleftrightarrow{I}\right]$$

$$+\left(\frac{-q_1}{q_1+q_2}\right)^2 q_2\left[3s^2\hat{z}\hat{z} - s^2\overleftrightarrow{I}\right]$$

$$\overleftrightarrow{Q}' = \frac{q_2^2 q_1 + q_1^2 q_2}{(q_1+q_2)^2} s^2 \left[3\hat{z}\hat{z} - \overleftrightarrow{I} \right]$$

$$= \frac{q_1 q_2}{q_1+q_2} s^2 \left[3\hat{z}\hat{z} - \overleftrightarrow{I} \right]$$

$$Q'_{ij} = \frac{q_1 q_2}{q_1+q_2} s^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

in xyz coord system

as $\hat{z}\hat{z} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\overleftrightarrow{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

the contribution of quadrupole to the potential is

$$\Phi_{quad} = \frac{1}{2} \frac{\hat{r} \cdot \overleftrightarrow{Q} \cdot \hat{r}}{r^3}$$

$\hat{r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$ in xyz coords

with origin at O' this becomes

$$\Phi_{quad} = \frac{s^2}{2r^3} \frac{q_1 q_2}{q_1+q_2} (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

do matrix multiplications

$$\Phi_{quad} = \frac{s^2}{2r^3} \frac{q_1 q_2}{q_1+q_2} (2\cos^2\theta - \sin^2\theta)$$

independent of φ as it must be due to azimuthal symmetry

Example

sample charge config's

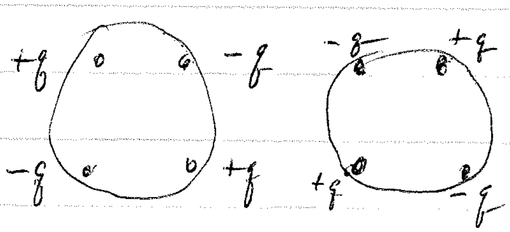
$\circ\ q$ $\Rightarrow$ monopole is leading term

$+q \quad -q$ $\Rightarrow$ monopole $= 0$ $\Rightarrow$ dipole is leading term

$\vec{p}$ is indep of origin

$+q \quad -q$ $\Rightarrow$ monopole $= 0$ $\Rightarrow$ total dipole is
$-q \quad +q$ sum of dipoles of individual neutral pairs

$\leftarrow + \rightarrow = 0$ leading term is quadrapole



when monopole $= 0$ and dipole $= 0$, quadrapole is indep of origin.
$\rightarrow$ total quadrapole is sum of quadrapoles of individual clusters with $q = 0$ and $\vec{p} = 0$

$$Q = Q_1 + Q_2$$

with $Q_2 = -Q_1$

$\Rightarrow Q = 0$ leading term is <u>octopole</u>

Magnetostatics

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad , \quad \begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = \dfrac{4\pi}{c} \vec{j} \end{cases}$$ Ampere's Law (statics only!)

$$\Rightarrow \quad \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{4\pi}{c} \vec{j}$$

can write $\quad \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

where by $\nabla^2 \vec{A}$ we mean $(\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}$

$\nabla^2 \vec{A}$ only has a simple expression in Cartesian coords

Aside

If tried to write it in spherical coords, for example, one has

$$\nabla^2 \vec{A} = \nabla^2 (A_r \hat{r} + A_\theta \hat{\theta} + A_\varphi \hat{\varphi})$$

$$= \cancel{(\nabla^2 A_r)\hat{r} + A_r(\nabla^2 \hat{r}) + (\nabla^2 A_\theta)\hat{\theta} + A_\theta(\nabla^2 \hat{\theta})}$$
$$\cancel{+ (\nabla^2 A_\varphi)\hat{\varphi} + A_\varphi(\nabla^2 \hat{\varphi})}$$

one must not forget to take the derivates of $\hat{r}, \hat{\theta}, \hat{\varphi}$ since they vary with position!

for example, $\hat{r} = \sin\theta\cos\varphi \, \hat{x} + \sin\theta\sin\varphi \, \hat{y} + \cos\theta \, \hat{z}$

one could compute $\nabla^2 \hat{r}$ by applying ∇^2 in spherical coords to each piece and summing up. Get a mess!

If work in Coulomb gauge, with $\vec{\nabla} \cdot \vec{A} = 0$, then

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \boxed{-\nabla^2 \vec{A} = \frac{4\pi}{c} \vec{j}}$$ Poisson's equation!

Many of the same methods used to solve for electrostatic ϕ can therefore be applied to solve for magnetostatic $\vec{A}$. But vector nature of eqn makes for complications!

For simple geometries, one can do the Coulomb-like integral

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$ three equations for A_x, A_y, A_z !

for localized current sources $j(r) \to 0$ as $r \to \infty$

<u>Multipole expansion</u> - magnetic dipole moment

For a general treatment, analogous to how we did multipole expansion for electrostatics, one can use <u>vector spherical harmonics</u> - see Jackson Chpt 9.

Here we do a more straight forward approach, but only up to magnetic dipole term.

For $r \gg r'$ approx

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{(r^2 - 2\vec{r}\cdot\vec{r}' + r'^2)^{1/2}} = \frac{1}{r}\,\frac{1}{\left[1 - 2\frac{\vec{r}\cdot\vec{r}'}{r^2} + \left(\frac{r'}{r}\right)^2\right]^{1/2}}$$

do Taylor series to 1st order in $\left(\frac{r'}{r}\right)$ to get

$$\frac{1}{|\vec{r}-\vec{r}'|} \simeq \frac{1}{r}\left\{1 + \frac{\vec{r}\cdot\vec{r}'}{r^2} + \cdots\right\} = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \cdots$$

$$\vec{A}(\vec{r}) = \overset{①}{\int \frac{d^3r'}{c} \frac{\vec{j}(\vec{r}')}{r}} + \overset{②}{\int \frac{d^3r'}{c} \vec{j}(\vec{r}') \frac{(\vec{r}\cdot\vec{r}')}{r^3}} + \dots$$

<u>Consider term ①</u>

$$\int d^3r\, \vec{j}(\vec{r}) \qquad \int d^3r (\vec{j}\cdot\vec{\nabla})\vec{r} \qquad \frac{\partial r_i}{\partial r_j} = \delta_{ij}$$

write $\displaystyle\int d^3r\, j_i(r) = \sum_{j=1}^{3} \int d^3r\, j_j \frac{\partial r_i}{\partial r_j}$ integrate by parts

$$= \sum_j \left\{ \oint_S da\, j_j r_i - \int d^3r \frac{\partial j_j}{\partial r_j} r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{j}$ sufficiently localized ie $\vec{j}(\vec{r}) \to 0$ sufficiently fast as $r \to \infty$

vanishes in <u>magnetostatics</u> where $\vec{\nabla}\cdot\vec{j} = 0$

So $\displaystyle\int d^3r\, \vec{j}(\vec{r}) = 0$ in magnetostatics

monopole term vanishes

term ②

$$\int d^3r \; \vec{J}(\vec{r})\,\vec{r} \qquad \text{tensor}$$

Consider $\int d^3r \; j_i r_j = \sum_k \int d^3r \; j_k r_j \frac{\partial r_i}{\partial r_k}$ integrate by parts

$$= \sum_k \left\{ \oint_S da \; j_k r_j r_i - \int d^3r \frac{\partial}{\partial r_k}(j_k r_j)\, r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{J}$ sufficiently localized

$$= -\sum_k \int d^3r \left(\frac{\partial j_k}{\partial r_k} r_j r_i + j_k \frac{\partial r_j}{\partial r_k} r_i \right)$$

vanishes as $\vec{\nabla}\cdot\vec{J} = 0$ in magnetostatics $\qquad = \delta_{jk}$

$$= -\int d^3r \; j_j r_i$$

So $\int d^3r \; j_i r_j = -\int d^3r \; j_j r_i$

$$= \frac{1}{2}\int d^3r \left(j_i r_j - j_j r_i \right)$$

So

$$\int d^3r' \, j_i(\vec{r}')(\vec{r}\cdot\vec{r}') = \sum_j r_j \int d^3r' \, j_i(\vec{r}')\, r_j'$$

$$= \sum_j \frac{1}{2}\int d^3r' \left(j_i r_j r_j' - r_j j_j r_i' \right)$$

$$= \frac{1}{2}\int d^3r' \left(j_i (\vec{r}\cdot\vec{r}') - r_i' (\vec{r}\cdot\vec{J}) \right)$$

use triple product rule

$$\vec{r} \times (\vec{r}' \times \vec{J}) = \vec{r}' (\vec{r} \cdot \vec{J}) - \vec{J} (\vec{r} \cdot \vec{r}')$$

to rewrite as

$$\int d^3r' \, \vec{J} \, (\vec{r} \cdot \vec{r}') = -\frac{1}{2} \vec{r} \times \left[\int d^3r' \, \vec{r}' \times \vec{J}(\vec{r}') \right]$$

define the magnetic dipole moment as

$$\boxed{\vec{m} = \frac{1}{2c} \int d^3r' \, \vec{r}' \times \vec{J}(\vec{r}')}$$

In magnetic dipole approx (this is the lowest non-vanishing term)

$$\vec{A}(\vec{r}) = -\frac{\vec{r} \times \vec{m}}{r^3} = \frac{\vec{m} \times \vec{r}}{r^3} = \frac{\vec{m} \times \hat{r}}{r^2}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(\vec{m} \times \frac{\vec{r}}{r^3} \right)$$

$$B_i = \varepsilon_{ijk} \, \partial_j \, \varepsilon_{klm} \, m_l \frac{r_m}{r^3}$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \, \partial_j \, m_l \frac{r_m}{r^3}$$

$$= m_i \, \partial_j \left(\frac{r_j}{r^3} \right) - m_j \partial_j \left(\frac{r_i}{r^3} \right)$$

$$= m_i \left[-4\pi \delta(\vec{r}) \right] - m_j \left[\frac{\delta_{ij}}{r^3} - \frac{3 r_i}{r^4} \partial_j r \right]$$

$$= \underset{\text{far from source}}{0} - \frac{m_i}{r^3} + 3 \frac{r_i}{r^4} \frac{r_j}{r} m_j$$

Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{any two indices equal} \\ +1, & ijk \text{ even permutat. of } 123 \\ -1, & ijk \text{ odd permutat. } 123 \end{cases}$$

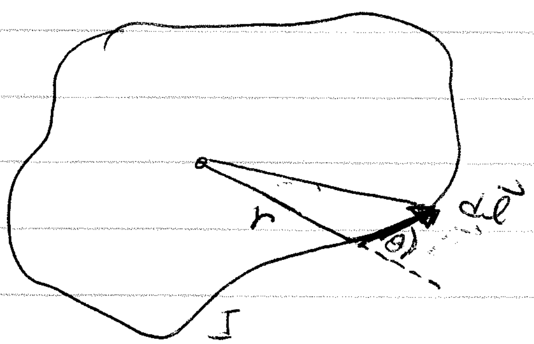
$$(\vec{A} \times \vec{B})_i = \varepsilon_{ijk} A_j B_k$$

$$\boxed{\vec{B} = \frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3}}$$

same form as $\vec{E}$ from electric dipole $\vec{p}$

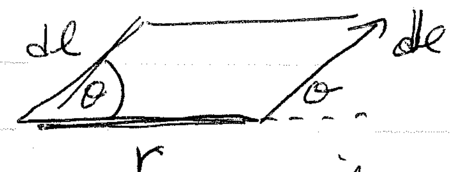
For a current loop in a plane (any shape loop provided it is flat)

$$\vec{m} = \frac{1}{2c}\int d^3r\ \vec{r}\times\vec{j} = \frac{1}{2c} I \oint \vec{r}\times d\vec{l}$$



area of triangle is $\frac{1}{2} r\, dl \sin\theta$

$= \frac{1}{2} |\vec{r}\times d\vec{l}|$



area of ~~rectangle~~ trapazoid is $r\, dl \sin\theta$

$$\Rightarrow \quad \vec{m} = \frac{1}{c} I(\text{area})\,\hat{n}$$

area of loop

outward normal (direction given by right hand rule with respect to direction of current) magnetic dipole moment $\vec{m}$ is independent of location of origin.

$$\vec{r}\,' = \vec{r} + \vec{d} \quad \text{new coord}$$

$$\vec{m}\,' = \frac{1}{2c}\int d^3r' \,(\vec{r}\,' \times \vec{j}\,) = \frac{1}{2c}\int d^3r \,(\vec{r} + \vec{d}\,) \times \vec{j}$$

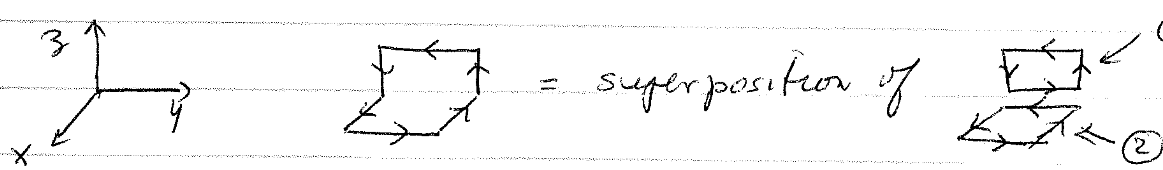
$$= \frac{1}{2c}\int d^3r \,\vec{r} \times \vec{j} + \frac{1}{2c}\vec{d} \times \left[\int d^3r \,\vec{j}\,\right]$$

$$\vec{m}\,' = \vec{m} + 0 \quad \text{as} \quad \int d^3r \,\vec{j} = 0$$

---

for planar loop $\vec{m} = \frac{I}{c} a \hat{n}$ where a = area
$\hat{n}$ = outward normal

Can also apply to get $\vec{m}$ for <u>piecewise planar loops</u>



$$\vec{m} = \vec{m}_1 + \vec{m}_2 \qquad \vec{m}_1 = \frac{I}{c} a_1 \hat{x}$$
$$\vec{m}_2 = \frac{I}{c} a_2 \hat{z}$$

$$\Rightarrow \quad \vec{m} = \frac{I}{c}(a_1 \hat{x} + a_2 \hat{z})$$