

## Boundary value problems in magnetostatics

### Scalar Magnetic Potential

Because of the vector character of the equation

$$-\nabla^2 \vec{A} = \frac{4\pi}{c} \vec{j}$$

and the fact that  $\nabla^2 \vec{A}$  only has a convenient representation in Cartesian coordinates, many of the methods we used to solve the scalar  $-\nabla^2 \phi = 4\pi\rho$  don't work so well for magnetostatics.

However, in situations where the current  $\vec{j}$  is confined to certain surfaces, we can make things much closer to the electrostatic case by using the trick of the scalar magnetic potential  $\phi_M$ .

In regions where  $\vec{j} = 0$ , i.e. not on the certain surfaces, we have  $\vec{\nabla} \cdot \vec{B} = 0$  and  $\vec{\nabla} \times \vec{B} = 0$ . Since  $\vec{\nabla} \times \vec{B} = 0$  in these regions, we can define a scalar potential  $\phi_M$  such that

$$\vec{B} = -\vec{\nabla} \phi_M$$

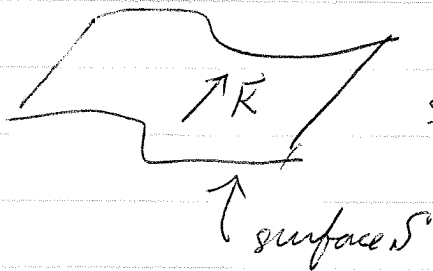
and then

$$\vec{\nabla} \cdot \vec{B} = -\nabla^2 \phi_M = 0$$

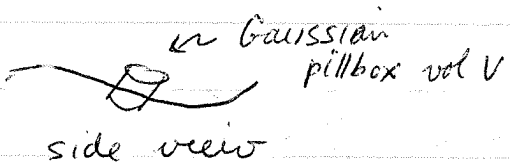
We can solve for  $\phi_M$  as in electrostatics, and match solutions by applying appropriate boundary conditions on the current carrying surfaces.

# Boundary Conditions at Sheet Current

in magnetostatics  $\vec{\nabla} \cdot \vec{B} = 0$ ,  $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$



surface current  $\vec{K}(\vec{r})$  at pt  $\vec{r}$   
on surface  $S$



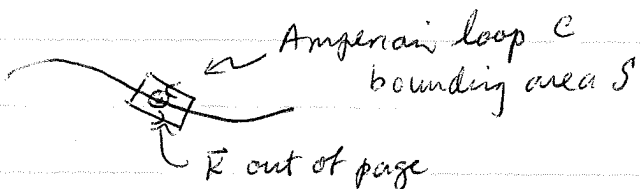
Gaussian pillbox vol  $V$

$$\int_V d^3r \vec{\nabla} \cdot \vec{B} = 0$$

top + bottom area of pill box is  $da$   
width of pill box  $\rightarrow 0$

$$\Rightarrow \int_V d^3r \vec{\nabla} \cdot \vec{B} = \oint_S da \hat{n} \cdot \vec{B} = da (\vec{B}_{\text{above}} - \vec{B}_{\text{below}}) \cdot \hat{n} = 0$$

normal component of  $\vec{B}$  is continuous  $(\vec{B}_{\text{above}} - \vec{B}_{\text{below}}) \cdot \hat{n} = 0$



Amperian loop  $c$

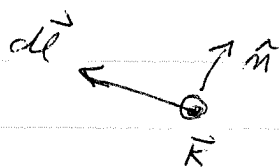
bounding area  $S$

$\vec{K}$  out of page

side view

$$\int_S da \hat{n} \cdot (\vec{\nabla} \times \vec{B}) = \oint_c \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I_{\text{enclosed}}$$

let width of loop  $\rightarrow 0$ , top + bottom sides  $d\vec{l}$



$$(\hat{n} \times d\vec{l}) \cdot \vec{K}$$

$$(\vec{B}_{\text{above}} - \vec{B}_{\text{below}}) \cdot d\vec{l} = \frac{4\pi}{c} K dl$$

$$= \frac{4\pi}{c} (\vec{K} \times \hat{n}) \cdot d\vec{l}$$

$\hat{n}$  is outward normal

tangential component of  $\vec{B}$  has

discontinuous jump  $\frac{4\pi}{c} \vec{K} \times \hat{n}$

Combine both results into

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \frac{4\pi}{c} \vec{K} \times \hat{n}$$

magnetic analog of  $\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = 4\pi\sigma \hat{n}$

In terms of magnetic ~~vector~~<sup>scalar</sup> potential  $\phi_M$

$$-\vec{\nabla}\phi_{M\text{above}} + \vec{\nabla}\phi_{M\text{below}} = \frac{4\pi}{c} \vec{K} \times \hat{n}$$

Note:  $\phi_M$  is a calculational tool only  
it does not have any direct physical significance as does the electrostatic  $\phi$ .  
Electrostatic  $\phi$  is related to work done moving a charge  $W_{12} = q[\phi(r_2) - \phi(r_1)]$   
nothing similar for  $\phi_M$ .

(in fact magnetostatic magnetic forces do no work!)

$$\vec{F} = q \vec{v} \times \vec{B}$$
$$\Rightarrow \vec{F} \cdot \vec{v} = \frac{dW}{dt} = 0$$

Note: We cannot apply argument <sup>like</sup>  $\phi(r) - \phi(R) = \int_r^R \vec{E} \cdot d\vec{l}$   
 $\phi_M$  is not necessarily continuous at surface  $r'$  current  
Cannot do similar to electrostatics and use

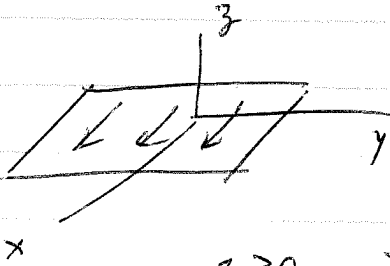
$$\phi_M(r_{\text{above}}) - \phi_M(r_{\text{below}}) = - \int_{r_{\text{below}}}^{r_{\text{above}}} \vec{B} \cdot d\vec{l}$$

Since  $\phi_M$  is not defined on the current sheet itself, separating "above" from "below".

example

Flat infinite plane at  $z=0$  with surface current

$$\vec{K} = K \hat{x}$$



$$z \geq 0, \quad \nabla^2 \Phi_M^> = 0 \Rightarrow \Phi_M^> = a^> - b_x^> x - b_y^> y - b_z^> z$$
$$z < 0, \quad \nabla^2 \Phi_M^< = 0 \Rightarrow \Phi_M^< = a^< - b_x^< x - b_y^< y - b_z^< z$$

$$z \geq 0, \quad \vec{B}^> = -\vec{\nabla} \Phi_M^> = b_x^> \hat{x} + b_y^> \hat{y} + b_z^> \hat{z}$$
$$z < 0, \quad \vec{B}^< = -\vec{\nabla} \Phi_M^< = b_x^< \hat{x} + b_y^< \hat{y} + b_z^< \hat{z}$$

$$\text{at } z=0 \quad \vec{B}^> - \vec{B}^< = (b_x^> - b_x^<) \hat{x} + (b_y^> - b_y^<) \hat{y} + (b_z^> - b_z^<) \hat{z}$$
$$= \frac{4\pi K}{c} \hat{x} \times \hat{z} = \frac{4\pi K}{c} (\hat{x} \times \hat{z}) = -\frac{4\pi K}{c} \hat{y}$$

$$\Rightarrow b_x^> = b_x^< \equiv b_{x0}, \quad b_z^> = b_z^< \equiv b_{z0}, \quad b_y^> - b_y^< = -\frac{4\pi K}{c}$$

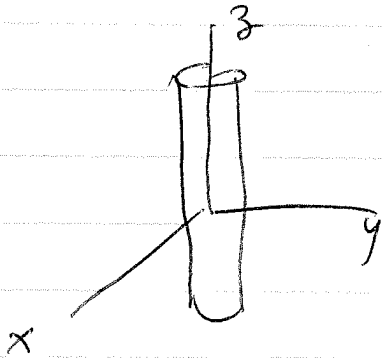
$$\text{define } \left. \begin{aligned} b_y^> &= b_{y0} + \delta b_y \\ b_y^< &= b_{y0} - \delta b_y \end{aligned} \right\} \delta b_y = -\frac{2\pi K}{c}$$

$$\Rightarrow \vec{B}^> = \vec{B}_0 - \frac{2\pi K}{c} \hat{y} \quad \vec{B}_0 = b_{x0} \hat{x} + b_{y0} \hat{y} + b_{z0} \hat{z}$$
$$\vec{B}^< = \vec{B}_0 + \frac{2\pi K}{c} \hat{y}$$

if  $\vec{K}$  is the only source of magnetic field then  $\vec{B}_0 = 0$

$$\vec{B} = \begin{cases} -\frac{2\pi K}{c} \hat{y} & z \geq 0 \\ \frac{2\pi K}{c} \hat{y} & z < 0 \end{cases}$$

example current carrying infinite cylinder radius  $R$



- (i)  $\vec{K} = K \hat{z}$  wire with surface current  
 (ii)  $\vec{K} = K \hat{\phi}$  solenoid

(i)  $\vec{K} = K \hat{z}$

$2\pi R K = I$  total current  
 "guess" + show it is correct

$r > R$   $\Phi_M = -\frac{4\pi R K}{c} \varphi$  magnetic scalar potential  $\nabla^2 \Phi_M = 0$   
 $r < R$   $\Phi_M = 0$

$r > R$   $\vec{B} = -\vec{\nabla} \Phi_M = -\frac{1}{r} \frac{\partial \Phi_M}{\partial \varphi} \hat{\phi} = \frac{4\pi R K}{c r} \hat{\phi} = \frac{2I}{c r} \hat{\phi}$  ← further result from Ampere  
 $r < R$   $\vec{B} = 0$

$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \frac{2I}{cR} \hat{\phi} = \frac{4\pi K}{c} \frac{R}{R} \hat{\phi} = \frac{4\pi \vec{K}}{c} \times \hat{n}$   
 where  $\hat{n} = \hat{r}$   
 as  $\hat{z} \times \hat{r} = \hat{\phi}$

Note:  $\Phi_M = -\frac{4\pi R K}{c} \varphi$  is not single valued!

would not have found this using expansion of separation of coords in polar coords

$\Phi_M$  does not need to be single valued since it has no physical significance, only  $\vec{B} = -\vec{\nabla} \Phi_M$  is physical

(ii)  $\vec{K} = K \hat{\phi}$

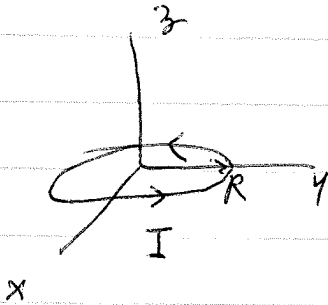
$r > R$   $\Phi_M = -B_1 z$   
 $r < R$   $\Phi_M = -B_2 z$  }  $\nabla^2 \Phi_M = 0$   
 $r > R$   $\vec{B} = -\vec{\nabla} \Phi_M = B_1 \hat{z}$   
 $r < R$   $\vec{B} = -\vec{\nabla} \Phi_M = B_2 \hat{z}$

$$\begin{aligned}
 \vec{B}_{\text{above}} - \vec{B}_{\text{below}} &= (B_1 - B_2) \hat{z} = \frac{4\pi}{c} \vec{K} \times \hat{n} \\
 &= \frac{4\pi}{c} K (\hat{\phi} \times \hat{r}) \\
 &= -\frac{4\pi}{c} K \hat{z}
 \end{aligned}$$

If current in solenoid is only source of  $\vec{B}$  then expect  $B_1 = 0$

$$\Rightarrow \boxed{\vec{B}_2 = \frac{4\pi}{c} K \hat{z}} \quad \text{familiar result}$$

example circular current loop in  $xy$  plane  
radius  $R$



for  $r > R$ ,  $\vec{\nabla} \times \vec{B} = 0 \Rightarrow \vec{B} = -\vec{\nabla} \phi_M$   
where  $\nabla^2 \phi_M = 0$ .

Try Legendre polynomial expansion for  $\phi_M$

$$\phi_M = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \quad (A_l \text{ terms vanish as want } B \rightarrow 0 \text{ as } r \rightarrow \infty)$$

$$\vec{B} = -\vec{\nabla} \phi_M = -\frac{\partial \phi_M}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial \phi_M}{\partial \theta} \hat{\theta}$$

$$= \sum_l \left[ \frac{(l+1)B_l}{r^{l+2}} P_l(\cos\theta) \hat{r} - \frac{B_l}{r^{l+2}} \frac{\partial P_l(\cos\theta)}{\partial \theta} \hat{\theta} \right]$$

write  $\frac{\partial P_l}{\partial \theta} = \frac{\partial P_l}{\partial x} \frac{\partial x}{\partial \theta} = -\frac{\partial P_l}{\partial x} \sin\theta \quad x = \cos\theta$   
 $\equiv -P_l' \sin\theta$

$$\vec{B} = \sum_l \left[ \frac{(l+1)B_l}{r^{l+2}} P_l(\cos\theta) \hat{r} + \frac{B_l}{r^{l+2}} \sin\theta P_l'(\cos\theta) \hat{\theta} \right]$$

To determine the  $B_l$  we compare with exact solution along  $\hat{z}$  axis

$$\vec{B}(z\hat{z}) = \sum_l \frac{(l+1)B_l}{r^{l+2}} \hat{r} = \sum_l \frac{(l+1)B_l}{z^{l+2}} \hat{z}$$

since  $P_l(1)=1$ ,  $\sin(0)=0$  and  $P_l'(1)$  finite,  $\hat{r} = \hat{z}$   
"the.  $r = z$

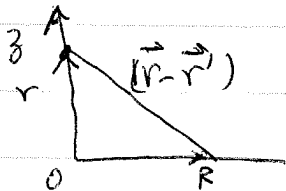
exact solution on  $\hat{z}$  axis:

$$\vec{A} = \int \frac{d^3r'}{c} \frac{\vec{j}(r')}{|\vec{r}-\vec{r}'|} \Rightarrow \vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A} = \int \frac{d^3r'}{c} \vec{\nabla} \times \frac{\vec{j}(r')}{|\vec{r}-\vec{r}'|}$$

$$\vec{B} = - \int \frac{d^3r'}{c} \vec{j}(r') \times \vec{\nabla} \left( \frac{1}{|\vec{r}-\vec{r}'|} \right)$$

$$\vec{B} = \int \frac{d^3r'}{c} \vec{j}(r') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \quad \text{Biot-Savart Law for magnetostatics}$$

For our loop



$$\vec{B}(z) = \int_0^{2\pi} d\phi \frac{R}{c} I \hat{\phi} \times \frac{[-R \hat{r} + z \hat{z}]}{(z^2 + R^2)^{3/2}}$$

$$\hat{r} \times \hat{\phi} = \hat{z}$$

$$= \int_0^{2\pi} \frac{d\phi}{c} \frac{R(I R) \hat{z}}{(z^2 + R^2)^{3/2}}$$

$\hat{\phi} \times \hat{z}$  term integrates to zero

$$\vec{B}(z) = \frac{2\pi R^2 I \hat{z}}{c (z^2 + R^2)^{3/2}}$$

to match Legendre polynomial expansion, do Taylor series expansion of above

$$\vec{B}(z) = \frac{2\pi R^2 I \hat{z}}{c z^3} \frac{1}{\left(1 + \left(\frac{R}{z}\right)^2\right)^{3/2}} = \frac{2\pi R^2 I \hat{z}}{c z^3} \left\{ 1 - \frac{3}{2} \left(\frac{R}{z}\right)^2 + \dots \right\}$$

$$= \frac{2\pi R^2 I \hat{z}}{c} \left\{ \frac{1}{z^3} - \frac{3}{2} \frac{R^2}{z^5} + \dots \right\}$$

$$= \left\{ \frac{B_0}{z^2} + \frac{2B_1}{z^3} + \frac{3B_2}{z^4} + \frac{4B_3}{z^5} + \dots \right\} \hat{z}$$



$$\Rightarrow B_0 = 0, \quad B_1 = \frac{\pi R^2 I}{c}, \quad B_2 = 0, \quad B_3 = -\frac{3}{4c} \pi R^2 I R^2$$

So to order  $l=3$

$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c} \left\{ \frac{2 P_1(\cos\theta) \hat{r} + \sin\theta P_1'(\cos\theta) \hat{\theta}}{r^3} - \left[ \frac{3 R^2 P_3(\cos\theta) \hat{r} + \frac{3}{4} R^2 \sin\theta P_3'(\cos\theta) \hat{\theta}}{r^5} \right] + \dots \right\}$$

$$P_1(x) = x \Rightarrow P_1'(x) = 1$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow P_3'(x) = \frac{1}{2}(15x^2 - 3)$$

$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c} \left\{ \frac{2 \cos\theta \hat{r} + \sin\theta \hat{\theta}}{r^3} - \left[ \frac{\frac{3}{2} R^2 (5 \cos^3\theta - 3 \cos\theta) \hat{r} + \frac{3}{8} R^2 \sin\theta (15 \cos^2\theta - 3) \hat{\theta}}{r^5} \right] + \dots \right\}$$

$\frac{\pi R^2 I}{c} = m$  is the magnetic dipole moment of the loop

We see that the 1st term is just the magnetic dipole approx. The 2nd term is the magnetic <sup>octapole</sup> ~~quadrupole~~ term. Could easily get higher order terms by this method.

Compare our result above to Jackson (5.40)