Then $\vec{\nabla}\cdot\vec{b} = 0 \Rightarrow \langle \vec{\nabla}\cdot\vec{b} \rangle = 0$

$\Rightarrow \vec{\nabla}\cdot\langle\vec{b}\rangle = 0$

$\Rightarrow \vec{\nabla}\cdot\vec{B} = 0$

$$\vec{\nabla}\times\vec{e} + \frac{\partial\vec{b}}{\partial t} = 0 \Rightarrow \vec{\nabla}\times\langle\vec{e}\rangle + \frac{\partial}{\partial t}\langle\vec{b}\rangle = 0$$

$$\Rightarrow \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

Renaming Maxwell eqns, upon averaging, become

$$\vec{\nabla}\cdot\vec{E} = 4\pi\langle\rho_0\rangle$$
$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\langle\vec{j}_0\rangle + \frac{\partial\vec{E}}{\partial t}$$

Consider $\langle\rho_0\rangle$

$\rho_0 = \sum_i q_i\,\delta(\vec{r} - \vec{r}_i(t))$ sum over all charges

Consider dividing the charge into "free" charges and "bound" charges, where the latter are associated with the molecules that make up the dielectric

$\rho_{free} = \sum_{i\ free} q_i\,\delta(\vec{r} - \vec{r}_i(t))$ sum over only free charges

$\rho_{bound} = \sum_n \rho_n(\vec{r},t)$

↑ charge distribution of molecule n

$\rho_n = \sum_{i\in n} q_i\,\delta(\vec{r} - \vec{r}_i(t))$ sum over charges in molecule n

$$\langle \rho_n(\vec{r},t)\rangle = \int d^3r'\, f(\vec{r}')\, \rho_n(\vec{r}-\vec{r}',t)$$

$$= \sum_{i\in n} q_i \int d^3r'\, f(\vec{r}')\, \delta(\vec{r}-\vec{r}' - \vec{r}_i(t))$$

$$= \sum_{i\in n} q_i\, f(\vec{r}-\vec{r}_i(t))$$

write $\vec{r}_i(t) = \vec{r}_n(t) + \vec{r}_{ni}(t)$

↑ position of center of ~~mass~~ mass of molecule n

↑ position of charge i of molecule n with respect to center of ~~mass~~ mass

$$\langle \rho_n(\vec{r},t)\rangle = \sum_{i\in n} q_i\, f(\vec{r}-\vec{r}_n-\vec{r}_{ni})$$

Since the $|\vec{r}_{ni}|$ are all of atomic length scale, and f is slowly varying on the length scale, we can expand

$$\langle \rho_n(\vec{r},t)\rangle = \sum_{i\in n} q_i \left[f(\vec{r}-\vec{r}_n) - \left(\vec{\nabla} f(\vec{r}-\vec{r}_n)\right)\cdot \vec{r}_{ni} + \frac{1}{2}\sum_{\alpha\beta=1}^{3} \frac{\partial f(\vec{r}-\vec{r}_n)}{\partial r_\alpha \partial r_\beta} (\vec{r}_{ni})_\alpha (\vec{r}_{ni})_\beta + \cdots \right]$$

$$= f(\vec{r}-\vec{r}_n)\left[\sum_{i\in n} q_i\right]$$

$$- \left(\vec{\nabla} f(\vec{r}-\vec{r}_n)\right)\cdot \sum_{i\in n} q_i \vec{r}_{ni}$$

$$+ \sum_{\alpha,\beta=1}^{3} \left(\frac{1}{6}\frac{\partial f(\vec{r}-\vec{r}_n)}{\partial r_\alpha \partial r_\beta}\right) 3\sum_{i\in n} q_i (r_{ni})_\alpha (r_{ni})_\beta$$

Define $q_n = \sum_{i \in n} q_i$ total charge molecule n

$\vec{p}_n = \sum_{i \in n} q_i \vec{r}_{ni}$ dipole moment about center of mass of molec n

$\overleftrightarrow{Q}'_n = \sum_{i \in n} 3 q_i \vec{r}_{ni} \vec{r}_{ni}$ quadrapole moment about center of mass of molec n

(prime on $\tilde{Q}'$ since definition here is a little different from that of multipole exp)

$$\langle \rho_n(\vec{r},t) \rangle = q_n f(\vec{r}-\vec{r}_n) - \vec{p}_n \cdot \vec{\nabla} f(\vec{r}-\vec{r}_n)$$

$$+ \frac{1}{6} \sum_{\alpha\beta} (Q'_n)_{\alpha\beta} \frac{\partial^2 f(\vec{r}-\vec{r}_n)}{\partial r_\alpha \partial r_\beta}$$

Now use $\langle \delta(\vec{r}-\vec{r}_n) \rangle = f(\vec{r}-\vec{r}_n)$ by definition of averaging

$$\Rightarrow \langle \rho_n(\vec{r},t) \rangle = \langle q_n \delta(\vec{r}-\vec{r}_n) \rangle$$

$$- \vec{\nabla} \cdot \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle \qquad \vec{\nabla} \cdot p\delta = \vec{p} \cdot \vec{\nabla}\delta$$

$$+ \frac{1}{6} \sum_{\alpha\beta} \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \langle (Q'_n)_{\alpha\beta} \delta(\vec{r}-\vec{r}_n) \rangle$$

Now

$$\langle \rho_{bound}(\vec{r},t) \rangle = \sum_n \langle \rho_n(\vec{r},t) \rangle$$

$$= \left\langle \sum_n q_n \delta(\vec{r}-\vec{r}_n) \right\rangle - \vec{\nabla} \cdot \left\langle \sum_n \vec{p}_n \delta(\vec{r}-\vec{r}_n) \right\rangle$$

$$+ \frac{1}{6} \sum \frac{\partial^2}{\qquad} \left\langle \sum_n (\qquad \delta(\vec{r}-\vec{r}_n) \right\rangle$$

Define $\vec{\mathbb{P}}(\vec{r},t) \equiv \left\langle \sum_n \vec{p}_n \delta(\vec{r}-\vec{r}_n) \right\rangle$ average polarization density

$\overleftrightarrow{\mathbb{Q}}'(r,t) \equiv \frac{1}{6}\left\langle \sum_n \overleftrightarrow{Q}'_n \delta(\vec{r}-\vec{r}_n) \right\rangle$ average quadrupole density

$$\langle \rho_{bound} \rangle = \sum_n \langle q_n \delta(\vec{r}-\vec{r}_n) \rangle - \vec{\nabla}\cdot\vec{\mathbb{P}} + \sum_{\alpha\beta} \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \overleftarrow{\mathbb{Q}}'_{\alpha\beta}$$

~~Paste~~

Define the macroscopic charge density

$$\rho \equiv \left\langle \sum_{i \text{ free}} q_i \delta(\vec{r}-\vec{r}_i) \right\rangle + \left\langle \sum_{n \text{ molec}} q_n \delta(\vec{r}-\vec{r}_n) \right\rangle$$

Then

$$\vec{\nabla}\cdot\vec{E} = 4\pi\langle \rho_0 \rangle = 4\pi\left[\rho - \vec{\nabla}\cdot\vec{\mathbb{P}} + \sum_{\alpha\beta} \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \overleftarrow{\mathbb{Q}}'_{\alpha\beta}\right]$$

$$\sum_\alpha \frac{\partial}{\partial r_\alpha}\left[E_\alpha + 4\pi \mathbb{P}_\alpha - 4\pi \sum_\beta \frac{\partial}{\partial r_\beta} \mathbb{Q}'_{\alpha\beta} \right] = 4\pi\rho$$

Define <u>electric displacement vector</u>

$$D_\alpha = E_\alpha + 4\pi \mathbb{P}_\alpha - 4\pi \sum_\beta \frac{\partial}{\partial r_\beta} \mathbb{Q}'_{\alpha\beta}$$

Then $\boxed{\vec{\nabla}\cdot\vec{D} = 4\pi\rho}$

In most materials, the quadrupole and higher terms are negligible and we can take

$\vec{D} = \vec{E} + 4\pi\vec{\mathbb{P}}$

Why quadrupole + higher order terms can generally be ignored:

Let a_0 be the length scale that characterizes the size of a molecule in the dielectric
Let ℓ be the typical spacing between molecules
Let L be the length scale of the spatial averaging function $f(\vec{r})$. $L \gg a_0$

dipole moment per a_0

$\Rightarrow$ polarization $\mathbb{P} \sim \frac{a_0}{\ell^3}$

$$\vec{\nabla} \cdot \vec{\mathbb{P}} \sim \left(\frac{a_0}{L}\right) \frac{1}{\ell^3}$$

since $\mathbb{P}$ cannot vary on length scale shorter than the averaging length L

quadrupole moment

$$Q \sim a_0^2$$

quadrupole density $\mathbb{Q} \sim \frac{a_0^2}{\ell^3}$

$$\frac{\partial^2 \mathbb{Q}}{\partial r_\alpha \partial r_\beta} \sim \left(\frac{a_0}{L}\right)^2 \frac{1}{\ell^3}$$

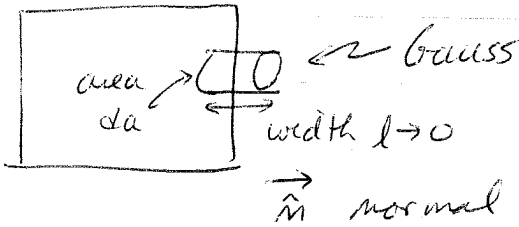
each higher moment gives extra factor a_0
each higher derivative gives extra factor $\frac{1}{L}$

So quadrupole is smaller than dipole term by factor $\left(\frac{a_0}{L}\right) \ll 1$. Higher terms smaller by additional factors of $\left(\frac{a_0}{L}\right)$

For insulators one generally has $\rho_{fr} = 0$
molecules are neutral.
In this case the macroscopic ρ is just the free charge $\rho = \langle \rho_{free} \rangle$.
And the bound charge is just

$$\boxed{\langle \rho_{bound} \rangle = -\vec{\nabla} \cdot \vec{P}}$$

At a surface of a dielectric



$$-\int_V d^3r \; \vec{\nabla} \cdot \vec{P} = -\oint_S da \; \hat{n} \cdot \vec{P}$$

contrib from sides $\to 0$ as $\ell \to 0$
contrib from outside surface $= 0$ as $P = 0$ outside

$$= \hat{n} \cdot \vec{P} \, da$$

only contrib is from inside surface ($\hat{n}$ is outward normal

$$= \int_V d^3r \; \rho_{bound}$$

as $\ell \to 0$, $\int_V d^3r \; \rho_{bound} \to \int da \; \sigma_{bound} = da \, \sigma_{bound}$ surface charge

$$\Rightarrow \boxed{\sigma_{bound} = \hat{n} \cdot \vec{P}} \text{ at surface of dielectric}$$

The

Magnetic Materials

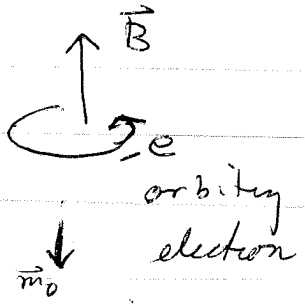
Circulating currents on atomic scale give rise to local magnetic dipole moments, which create local magnetic fields in the material.

Sources of circulating atomic currents:

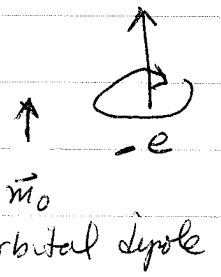
1) intrinsic angular momentum of electrons, ie "electron spin" - can add up and give a net angular momentum to atom
2) orbital angular momentum of electrons - can add up to give net angular momentum of atom.

(1) + (2) ⇒ atoms can have a net magnetic dipole moment. When $\vec{B}=0$, these atomic moments are generally in random orientations and average to zero (exception is a ferromagnet where moments can allign even if $\vec{B}=0$) When apply $\vec{B}\neq 0$, the moments tend to allign parallel to $\vec{B}$ giving a net magnetization density $\vec{M} \propto \vec{B}$. This is a paramagnetic effect.

But there is also a diamagnetic effect from orbital angular momentum (exists even if total angular momentum of electrons is zero, ie exists for atoms with zero net dipole moment)



← applying $\vec{B}$ to orbiting electron speeds up its orbital velocity. Increased angular momentum of negatively charged electron gives change in dipole moment $\Delta\vec{m} \propto -\vec{B}$



← applying $\vec{B}$ to orbiting electron slows down its orbital velocity. Net result is again that $\Delta\vec{m} \propto -\vec{B}$

see Griffiths chpt 6 + prob 7.17 2nd ed for details

No matter which way electron orbits with respect to $\vec{B}$, result is a decrease in magnetic moment, so $\Delta\vec{m} \propto -\vec{B}$. That $\Delta\vec{m}$ is opposite to $\vec{B}$ is called Lenz magnetism

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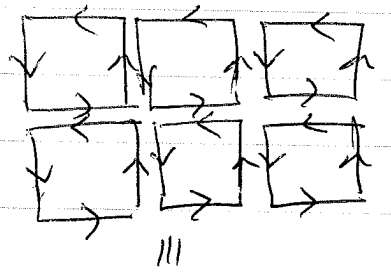
Model atomic magnetic moments as small current loops. When loops get oriented, ie there is non zero average magnetization density

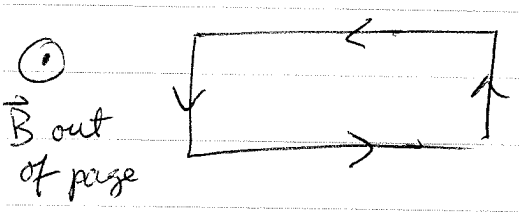
$$\vec{M}(\vec{r}) = \sum_i \vec{m}_i \, \delta(\vec{r} - \vec{r}_i)$$

then net effect is to have a current flowing around the system. This current gives rise to magnetic fields

aligned atomic moments in a uniform applied $\vec{B}$

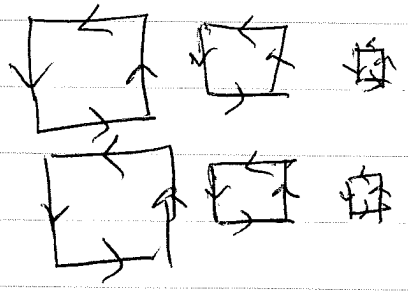


in interior, currents in opposite directions cancel also $\vec{j} = 0$ inside



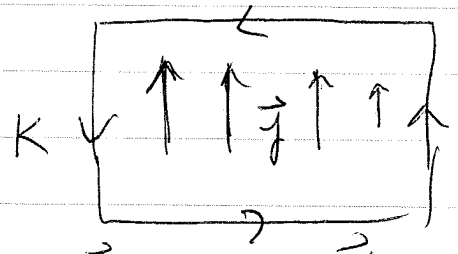
but is net circulation of current around boundary of material
$\Rightarrow$ surface current $\vec{K}_{bound}$

If $\vec{B}$ is not uniform, then $\vec{M}$ is not uniform
Can create finite current density $\vec{j}$ in interior, as well as surface currents



Now currents in interior do not cancel. Net current $\vec{j}_{bound}$ in interior

$\vec{B}$ strong $\vec{B}$ weak



$\vec{M}_{strong}$ $\vec{M}_{weak}$

$\vec{B}$ out of page $\Rightarrow$ $\vec{M}$ out of page
~~$\vec{M}$ varies along page~~
$\vec{M}$ varies in direction $\perp$ direction of $\vec{M}$
$\Rightarrow \vec{\nabla} \times \vec{M} \neq 0$ gives $\vec{j}_{bound}$

Average current

$$\langle \vec{j}_0 \rangle = \Big\langle \sum_{i \in \text{free}} q_i \vec{v}_i \, \delta(\vec{r} - \vec{r}_i) \Big\rangle + \sum_{\substack{n \\ \text{molec}}} \langle \vec{j}_n \rangle$$

↑ current from free charges

↑ current from molecule n of the dielectric

$$\langle \vec{j}_n(\vec{r}, t) \rangle = \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni}) \langle \delta(\vec{r} - \vec{r}_n(t) - \vec{r}_{ni}(t)) \rangle$$

$$= \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni}) \, f(\vec{r} - \vec{r}_n(t) - \vec{r}_{ni}(t))$$

↑ $\vec{v}_n = \frac{d\vec{r}_n}{dt}$ ↑ $\vec{v}_{ni} = \frac{d\vec{r}_{ni}}{dt}$ ↑ position of CM of molec n ↑ position of charge i wrt CM

as with $\langle \rho_0 \rangle$, we can expand in $\vec{r}_{ni}$

$$\langle \vec{j}_n \rangle = \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni}) \Big\{ f(\vec{r} - \vec{r}_n) - \vec{r}_{ni} \cdot \vec{\nabla} f(\vec{r} - \vec{r}_n)$$

$$+ \frac{1}{2} \sum_{\alpha\beta} (r_{ni})_\alpha (r_{ni})_\beta \frac{\partial^2 f(\vec{r} - \vec{r}_n)}{\partial r_\alpha \partial r_\beta}$$

$$+ \cdots \Big\}$$

we will keep only the first two terms in the expansion

The various terms we have to consider are

① $\sum_{i \in n} q_i \vec{v}_n f(\vec{r}-\vec{r}_n)$

② $\sum_{i \in n} q_i \vec{v}_{ni} \delta(\vec{r}-\vec{r}_n)$

③ $-\sum_{i \in n} q_i \vec{v}_n \left[\vec{r}_{ni} \cdot \vec{\nabla} f(\vec{r}-\vec{r}_n)\right]$

④ $-\sum_{i \in n} q_i \vec{v}_{ni} \left[\vec{r}_{ni} \cdot \vec{\nabla} f(\vec{r}-\vec{r}_n)\right]$

① $= \vec{v}_n f(\vec{r}-\vec{r}_n) \sum_{i \in n} q_i = q_n \vec{v}_n f(r-r_n)$

$= \langle q_n \vec{v}_n \delta(\vec{r}-\vec{r}_n) \rangle$

this is just current of molecule as if it were a point charge q_n. For a neutral molecule $q_n = 0$ and this term vanishes.

② Note: $\frac{\partial}{\partial t} \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle = \frac{\partial}{\partial t} \left(\sum_{i \in n} q_i \vec{r}_{ni} f(\vec{r}-\vec{r}_n) \right)$

$= \sum_{i \in n} q_i \vec{v}_{ni} f(\vec{r}-\vec{r}_n)$

$+ \sum_{i \in n} q_i \vec{r}_{ni} \left[-\vec{\nabla} f(\vec{r}-\vec{r}_n) \cdot \vec{v}_n\right]$

So for ②, $\sum_{i \in n} q_i \vec{v}_{ni} f(\vec{r}-\vec{r}_n)$

$= \frac{\partial}{\partial t} \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle$

$+ \left[\vec{v}_n \cdot \vec{\nabla} f(\vec{r}-\vec{r}_n)\right] \vec{p}_n$