\[ \nabla \cdot \mathbf{b} = 0 \quad \Rightarrow \quad \langle \nabla \cdot \mathbf{b} \rangle = 0 \]
\[ \Rightarrow \quad \nabla \cdot \langle \mathbf{b} \rangle = 0 \]
\[ \Rightarrow \quad \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \quad \nabla \times \langle \mathbf{E} \rangle + \frac{1}{c^2} \frac{\partial \langle \mathbf{B} \rangle}{\partial t} = 0 \]
\[ \Rightarrow \quad \nabla \times \langle \mathbf{E} \rangle + \frac{\partial \langle \mathbf{B} \rangle}{\partial t} = 0 \]

Riemann–Maxwell equations, upon averaging, become

\[ \nabla \cdot \mathbf{E} = 4\pi \langle \rho \rangle \]
\[ \nabla \times \mathbf{B} = \frac{4\pi}{c} \langle \mathbf{J} \rangle + \frac{\partial \mathbf{E}}{\partial t} \]

Consider \( \langle \rho \rangle \)

\[ \rho = \sum_i \delta_i \delta(r - r_i(t)) \quad \text{sum over all charges} \]

Consider dividing the charge into "free" charges and "bound" charges, where the latter are associated with the molecules that make up the dielectric.

\[ \rho_{\text{free}} = \sum_{i \text{ free}} \delta_i \delta(r - r_i(t)) \quad \text{sum over only free charges} \]

\[ \rho_{\text{bound}} = \sum_n \rho_n (r, t) \quad \text{charge distribution of molecule } n \]

\[ \rho_n = \sum_{i n} \delta_i \delta(r - r_i(t)) \quad \text{sum over charges in molecule } n \]
\[ \langle f_{n}(\vec{r},t) \rangle = \int d^{3}r' \ f(\vec{r}') \ \rho_{n}(\vec{r}-\vec{r}',t) \]

\[ = \sum_{\text{cen}} \ \delta_{i} \ \int d^{3}r' \ f(\vec{r}') \ \delta(\vec{r}-\vec{r}_i(t)) \]

\[ = \sum_{\text{cen}} \ \delta_{i} \ f(\vec{r}-\vec{r}_i(t)) \]

Write \( \vec{r}_i(t) = \vec{\rho}_n(t) + \vec{\rho}_n(t) \)

\[ \vec{r}_i(t) \]

\[ \text{position of charge } i \]

\[ \text{position of charge } j \]

\[ \text{center of mass of molecule } n \]

\[ \text{center of mass of molecule } n \]

\[ \text{center of mass of molecule } n \]

\[ \text{center of mass of molecule } n \]

Since the \( \vec{\rho}_n \) are all of atomic length scale, and \( f \) is slowly varying on the length scale, we can expand

\[ \langle f_{n}(\vec{r},t) \rangle = \sum_{\text{cen}} \ \delta_{i} \ [ f(\vec{r}-\vec{\rho}_n) - (\vec{\nabla} f(\vec{r}-\vec{\rho}_n)) \cdot \vec{\rho}_n + \frac{1}{2} \sum_{\alpha,\beta=1}^{3} \frac{\partial^{2} f(\vec{r}-\vec{\rho}_n)}{\partial r_{\alpha} \partial r_{\beta}} (\vec{\rho}_n)_{\alpha} (\vec{\rho}_n)_{\beta} + \cdots \]
Define \( q_n = \sum_{i \in n} q_i \) total charge molecule \( n \)

\[ \vec{p}_n = \sum_{i \in n} q_i \vec{r}_{ni} \] dipole moment about center of mass of molec. \( n \)

\[ \vec{Q}_n = \sum_{i \in n} 3 q_i \vec{r}_{ni} \vec{r}_{ni} \] quadrupole moment about center of mass of molec. \( n \)

(prime on \( \vec{r} \) since definition here is a little different from that of multipole exp.)

\[
\left< \vec{f}_n (\vec{r}, t) \right> = q_n \vec{f} (\vec{r} - \vec{r}_n) - \vec{p}_n \cdot \nabla \vec{f} (\vec{r} - \vec{r}_n) \\
+ \frac{1}{6} \sum_{\alpha \beta} (\vec{Q}_n')_{\alpha \beta} \frac{\nabla^2 \vec{f} (\vec{r} - \vec{r}_n)}{\hat{\nabla} \cdot \vec{r} \cdot \vec{r}_n}
\]

Now, use \( \left< \delta (\vec{r} - \vec{r}_n) \right> = \vec{f} (\vec{r} - \vec{r}_n) \) by definition of autangi.

\[ \Rightarrow \left< \vec{f}_n (\vec{r}, t) \right> = \left< q_n \delta (\vec{r} - \vec{r}_n) \right> - \nabla \cdot \left< \vec{p}_n \delta (\vec{r} - \vec{r}_n) \right> + \frac{1}{6} \sum_{\alpha \beta} \frac{\partial^2}{\partial \vec{r}_\alpha \partial \vec{r}_\beta} \left< (\vec{Q}_n')_{\alpha \beta} \delta (\vec{r} - \vec{r}_n) \right> 
\]

Now,

\[ \left< \vec{f}_{\text{bound}} (\vec{r}, t) \right> = \sum_n \left< \vec{f}_n (\vec{r}, t) \right> = \left< \sum_n q_n \delta (\vec{r} - \vec{r}_n) \right> - \nabla \cdot \left< \sum_n \vec{p}_n \delta (\vec{r} - \vec{r}_n) \right> + \frac{1}{6} \sum \frac{\partial^2}{\partial \vec{r}_\alpha \partial \vec{r}_\beta} \left< (\vec{Q}_n')_{\alpha \beta} \delta (\vec{r} - \vec{r}_n) \right> 
\]
Define \[ \overrightarrow{P}(\vec{r}, t) = \left\langle \sum_n \overrightarrow{p}_n \delta(\vec{r} - \vec{r}_n) \right\rangle \]

average polarizability density

\[ \overrightarrow{Q'}(\vec{r}, t) = \frac{1}{6} \left\langle \sum_n \overrightarrow{Q}'_n \delta(\vec{r} - \vec{r}_n) \right\rangle \]

average quadrupole density

\[ \left\langle \rho_{\text{bound}} \right\rangle = \sum_n \left\langle q_n \delta(\vec{r} - \vec{r}_n) \right\rangle - \nabla \cdot \overrightarrow{P} + \sum_{\alpha \beta} \frac{r^2}{\epsilon_\alpha \epsilon_\beta} \overrightarrow{Q}'_{\alpha \beta} \]

Define the macroscopic charge density

\[ \rho = \left\langle \sum_{c_{\text{free}}} q_c \delta(\vec{r} - \vec{r}_c) \right\rangle + \left\langle \sum_{n_{\text{molec}}} q_n \delta(\vec{r} - \vec{r}_n) \right\rangle \]

Then

\[ \nabla \cdot \overrightarrow{E} = 4\pi \rho = 4\pi \left[ \rho - \nabla \cdot \overrightarrow{P} + \sum_{\alpha \beta} \frac{r^2}{\epsilon_\alpha \epsilon_\beta} \overrightarrow{Q}'_{\alpha \beta} \right] \]

\[ \sum_A \frac{2}{\epsilon_A} \left[ E_A + 4\pi P_A - 4\pi Z \frac{2}{\epsilon_{\beta}} \overrightarrow{Q}'_{\alpha \beta} \right] = 4\pi \rho \]

Define electric displacement vector

\[ \overrightarrow{D}_A = E_A + 4\pi P_A - 4\pi Z \frac{2}{\epsilon_{\beta}} \overrightarrow{Q}'_{\alpha \beta} \]

then

\[ \nabla \cdot \overrightarrow{D} = 4\pi \rho \]

In most materials, the quadrupole and higher terms are negligible and we can take
Why quadrupole + higher order terms can generally be ignored?

Let $a_0$ be the length scale that characterizes the size of a molecule in the dielectric.
Let $l$ be the typical spacing between molecules.
Let $L$ be the length scale of the spatial averaging function $f(r)$. $L \gg a_0$

dipole moment $p \propto a_0$

\[ \Rightarrow \text{polarization} \ P \propto \frac{a_0}{L^3} \]

\[ \nabla \cdot \mathbf{P} \propto \left( \frac{a_0}{L} \right)^3 \] since $P$ cannot vary on length scale short than the averaging length $L$

quadrupole moment $Q \propto a_0^2$

quadrupole density $\mathbf{D} \propto \frac{a_0^2}{L^3}$

\[ \frac{\partial^2 \mathbf{D}}{\partial r_n \partial r_p} \propto \left( \frac{a_0}{L} \right)^2 \]

Each higher moment gives extra factor $a_0$
Each higher derivative gives extra factor $\frac{1}{L}$

so quadrupole is smaller than dipole term by factor $\left( \frac{a_0}{L} \right)^2 \ll 1$. Higher terms smaller by additional factors of $\left( \frac{a_0}{L} \right)$.
For insulators one generally has \( q_n = 0 \) molecules are neutral.

In this case the macroscopic \( \Phi \) is just the free charge \( \Phi = \langle \Phi_{\text{free}} \rangle \).

And the bound charge \( \Phi_{\text{bound}} \) is just

\[
\langle \Phi_{\text{bound}} \rangle = - \nabla \cdot \mathbf{P}
\]

At a surface of a dielectric

\[
\text{Gaussian pillbox} V
\]

\[
\begin{align*}
\text{area} & \quad \mathbf{C} \quad \text{Gaussian pillbox} V \\
\text{width} & \quad l \to 0 \\
\mathbf{n} & \quad \text{normal}
\end{align*}
\]

\[
- \int d^3 r \ \nabla \cdot \mathbf{P} = - \int_S d a \ \hat{\mathbf{n}} \cdot \mathbf{P}
\]

- contrib from sides \( \to 0 \) as \( l \to 0 \)
- contrib from outside surface \( = 0 \) as \( \mathbf{P} = 0 \) outside

\[
= \hat{\mathbf{n}} \cdot \mathbf{P} \ d a
\]

- only contrib \( \Phi \) from inside surface

\[
= \int d^3 r \ \Phi_{\text{bound}}
\]

- \( \hat{\mathbf{n}} \) outward normal

As \( l \to 0 \), \( \int d^3 r \ \Phi_{\text{bound}} \to \int d a \ \Phi_{\text{bound}} = d a \ \Phi_{\text{bound}} \) surface charge

\[
\Rightarrow \left\{ \Phi_{\text{bound}} = \hat{\mathbf{n}} \cdot \mathbf{P} \right\} \text{ at surface of dielectric}
\]
Magnetic Materials

Circulating currents on atomic scale give rise to local magnetic dipole moments, which create local magnetic fields in the material.

Sources of circulating atomic currents:
1) Intrinsic angular momentum of electrons, i.e., "electron spin" - can add up and give a net angular momentum to atom
2) Orbital angular momentum of electrons - can add up to give net angular momentum of atom.

(1) + (2) ⇒ atoms can have a net magnetic dipole moment. When \( \vec{B} = 0 \), these atomic moments can generally be in random orientations (exception is a ferromagnet where moments can align even if \( \vec{B} = 0 \)). When apply \( \vec{B} \neq 0 \), the moments tend to align parallel to \( \vec{B} \), giving a net magnetization density \( \vec{M} \propto \vec{B} \). This is a paramagnetic effect.

But there is also a diamagnetic effect from orbital angular momentum (exists even if total angular momentum of electrons is zero, i.e., exists for atoms with zero net dipole moment).
applying \( \mathbf{B} \) to an orbiting electron speeds up its orbital velocity. Increased angular momentum of negatively charged electron gives change in dipole moment \( \Delta \mathbf{m} \propto -\mathbf{B} \)

applying \( \mathbf{B} \) to orbiting electron slows down its orbital velocity. Net result is again that \( \Delta \mathbf{m} \propto -\mathbf{B} \)

No matter which way electron orbits with respect to \( \mathbf{B} \), result is a decrease in magnetic moment, so \( \Delta \mathbf{m} \propto -\mathbf{B} \)

That \( \Delta \mathbf{m} \) is opposite to \( \mathbf{B} \) is called deamagnetism

Model atomic magnetic moments as small current loops. When loops get oriented, ce there is non zero average magnetization density

\[
\mathbf{M}(\mathbf{r}) = \sum \mathbf{m}_i \delta(\mathbf{r} - \mathbf{r}_i)
\]

then net effect is to have a current flowing around the system. This current gives rise to magnetic fields
alleged atomic moments in a uniform applied $\vec{B}$

in interior, currents in opposite directions cancel also $\vec{J} = 0$ inside

but is net circulation of current around boundary of material $\Rightarrow \text{surface current } \vec{J}_{\text{bound}}$

If $\vec{B}$ is not uniform, then $\vec{M}$ is not uniform
Can create finite current density $\vec{J}$ in interior, as well as surface currents

Now current in interior do not cancel. Net current $\vec{J}_{\text{bound}}$ in interior

$\vec{B}$ strong $\quad \vec{B}$ weak

$\vec{B}$ out of page $\Rightarrow \vec{M}$ out of page
$\vec{M}$ varies in direction $\Rightarrow \vec{J}_{\text{bound}}$$\vec{M}_{\text{strong}}$ $\vec{M}_{\text{weak}}$$\Rightarrow \vec{J} \times \vec{M} \neq 0$ gives $\vec{J}_{\text{bound}}$
Average current

\[ \langle \vec{I}_c \rangle = \left\langle \sum_{i \text{ free}} g_i \vec{v}_i \delta (\vec{r} - \vec{r}_c) \right\rangle + \sum_{n} \left\langle \vec{I}_n \right\rangle \]

\[ \text{current from free charges} \]
\[ \text{molecule } n \]
\[ \text{current from molecule } n \text{ of the dielectric} \]

\[ < \vec{I}_n (\vec{r}, t) > = \sum_{i \in n} g_i (\vec{v}_n + \vec{v}_n) \left< \delta (\vec{r} - \vec{r}_n (t) - \vec{r}_n (t)) \right> \]

\[ = \sum_{i \in n} g_i (\vec{v}_n + \vec{v}_n) f (\vec{r} - \vec{r}_n (t) - \vec{r}_n (t)) \]

\[ \vec{v}_n = \frac{d\vec{r}_n}{dt} \quad \vec{v}_n = \frac{d\vec{r}_n}{dt} \quad \text{position of CM of molecule } n \]
\[ \text{position of charge } i \text{ wrt CM} \]

As with \( < I_0 > \), we can expand in \( \vec{r}_n \)

\[ < \vec{I}_n > = \sum_{i \in n} g_i (\vec{v}_n + \vec{v}_n) \int f (\vec{r} - \vec{r}_n) - \vec{r}_n \cdot \nabla f (\vec{r} - \vec{r}_n) \]
\[ + \frac{1}{2} \sum_{\alpha \beta} (\vec{r}_n)_{\alpha} (\vec{r}_n)_{\beta} \frac{\partial^2 f (\vec{r} - \vec{r}_n)}{\partial \alpha \partial \beta} \]

we will keep only the first two terms in the expansion.
The various terms we have to consider are:

1. \( \sum_{i \in \mathcal{N}} \mathbf{u}_i \cdot \mathbf{v}_n \cdot f(r-r_n) \)

2. \( \sum_{i \in \mathcal{N}} \delta_i \cdot \mathbf{v}_{ni} \cdot f(r-r_n) \)

3. \( -\sum_{i \in \mathcal{N}} \delta_i \cdot \mathbf{v}_n \cdot [\mathbf{r}_{ni} \cdot \nabla f(r-r_n)] \)

4. \( -\sum_{i \in \mathcal{N}} \delta_i \cdot \mathbf{v}_{ni} \cdot [\mathbf{r}_{ni} \cdot \nabla f(r-r_n)] \)

\[ 0 = \mathbf{v}_n + (r-r_n) \sum_{i \in \mathcal{N}} \mathbf{q}_i = \mathbf{q}_n \cdot \mathbf{v}_n \cdot f(r-r_n) \]

This is just current of molecule as if it were a point charge \( \mathbf{q}_n \). For neutral molecule \( \mathbf{q}_n = 0 \), the term vanishes.

Note:

\[ \frac{\partial}{\partial t} \langle \mathbf{p}_n \delta(r-r_n) \rangle = \frac{\partial}{\partial t} \left( \sum_{i \in \mathcal{N}} \mathbf{q}_i \mathbf{v}_{ni} \cdot f(r-r_n) \right) \]

\[ = \sum_{i \in \mathcal{N}} \delta_i \cdot \mathbf{v}_{ni} \cdot f(r-r_n) \]

\[ + \sum_{i \in \mathcal{N}} \delta_i \cdot \mathbf{v}_{ni} \cdot [\nabla f(r-r_n) \cdot \mathbf{v}_n] \]

So, for \( \mathbf{p}_n \),

\[ \sum_{i \in \mathcal{N}} \delta_i \cdot \mathbf{v}_{ni} \cdot f(r-r_n) \]

\[ = \frac{\partial}{\partial t} \langle \mathbf{p}_n \delta(r-r_n) \rangle \]

\[ + \left[ \mathbf{v}_n \cdot \nabla f(r-r_n) \right] \cdot \mathbf{p}_n \]