Linear Materials

Macroscopic Maxwell Equations

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \qquad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$

where ρ and $\vec{j}$ are macroscopic charge + current densities and

$$\vec{D} = \vec{E} + 4\pi\vec{P} \qquad \vec{P} \text{ is polarization density}$$
$$\vec{H} = \vec{B} - 4\pi\vec{M} \qquad \vec{M} \text{ is magnetization density}$$

To close these equations, we will in general need to know how $\vec{P}$ and $\vec{M}$ are related to the $\vec{E}$ and $\vec{B}$ in the material.

In <u>some</u> materials, there can be a finite $\vec{P}$ or $\vec{M}$ even if $\vec{E}$ and $\vec{B}$ are zero:

<u>Ferromagnet</u>: $\vec{M}$ can be non zero even if $\vec{B} = 0$

<u>Ferroelectric</u>: $\vec{P}$ can be non zero even if $\vec{E} = 0$

But more common are <u>linear</u> materials in which, for <u>small</u> $\vec{E}$ and $\vec{B}$, one has $\vec{P} \propto \vec{E}$ and $\vec{M} \propto \vec{B}$.

linear dielectric

$\vec{P} = \chi_e \vec{E}$ χ_e is "electric susceptibility"

$\chi_e > 0$ for statics

$\vec{D} = \vec{E} + 4\pi \vec{P} = (1 + 4\pi \chi_e) \vec{E}$

$\vec{D} = \epsilon \vec{E}$ with $\epsilon = 1 + 4\pi \chi_e$

ϵ is the dielectric constant

linear magnetic materials

$\vec{M} = \chi_m \vec{H}$ χ_m is "magnetic susceptibility"

$\chi_m > 0 \Rightarrow$ paramagnetic

$\chi_m < 0 \Rightarrow$ diamagnetic

$\vec{H} = \vec{B} - 4\pi \vec{M} = \vec{B} - 4\pi \chi_m \vec{H}$

$\vec{B} = (1 + 4\pi \chi_m) \vec{H}$

$\vec{B} = \mu \vec{H}$ with $\mu = 1 + 4\pi \chi_m$

μ is magnetic permeability

For statics, $\chi_e > 0$ and χ_m (or alternatively ϵ and μ) are constants depending on the material.

When we consider dynamics we will see that ϵ becomes a function of frequency.

Clausius - Mossotti equation

Electric susceptibility & atomic polarizability

If a field $\vec{E}_{loc}$ is applied to an atom, it gets polarized

$$\vec{p} = \alpha \vec{E}_{loc}$$

atomic dipole moment — atomic polarizability — "local field" - field the atom sees

α is what one calculates from a microscopic theory

If $\vec{E}_{loc} = \vec{E}$ the average field in the material then electric susceptibility given by

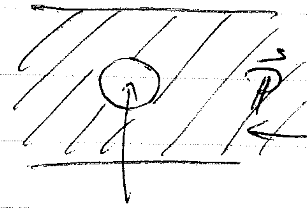
$$\vec{P} = n\vec{p} = n\alpha\vec{E}_{loc} = n\alpha\vec{E} = \chi_e \vec{E}$$

$\Rightarrow \chi_e = n\alpha$ where n = density of atoms

But a more careful consideration shows $\vec{E}_{loc} \neq \vec{E}$
The average field $\vec{E}$ includes the electric field created by the polarized atom itself. $\vec{E}_{loc}$, the local field the atom sees, should exclude its own self field.

$$\vec{E} = \vec{E}_{loc} + \vec{E}_{atom}$$

average field — average field excluding atom — average field of the atom



cut out sphere whose volume is $1/n$ the volume per atom

$\vec{E}_{loc}$ is field excluding the field of the polarized sphere of volume $1/n$.

$\vec{E}_{atom}$ is field of the polarized sphere

$$E_{atom} = -\frac{4\pi\vec{\mathbb{P}}}{3} = -\frac{4\pi}{3} n\vec{p}$$

$$\vec{E}_{loc} = \vec{E} - \vec{E}_{atom} = \vec{E} + \frac{4\pi}{3}\vec{\mathbb{P}} = \vec{E} + \frac{4\pi}{3} n\vec{p}$$

$$\vec{p} = \alpha\vec{E}_{loc} = \alpha\left(\vec{E} + \frac{4\pi}{3} n\vec{p}\right) = \alpha\vec{E} + \frac{4\pi}{3} n\alpha\,\vec{p}$$

$$\vec{p} = \frac{\alpha}{1 - \frac{4\pi}{3} n\alpha}\vec{E}$$

$$\vec{\mathbb{P}} = n\vec{p} = \frac{\alpha n}{1 - \frac{4\pi}{3} n\alpha}\vec{E} = \chi_e\vec{E}$$

$$\boxed{\chi_e = \frac{n\alpha}{1 - \frac{4\pi}{3} n\alpha}}$$

or solve for α in terms of ϵ

$$\chi_e = \frac{n\alpha}{1-\frac{4\pi}{3}n\alpha} \Rightarrow \chi_e - \frac{4\pi}{3}n\alpha\chi_e = \alpha n$$

$$\Rightarrow \alpha = \frac{\chi_e}{n(1+\frac{4\pi}{3}\chi_e)}$$

$$\epsilon = 1+4\pi\chi_e \Rightarrow \alpha = \frac{\epsilon-1}{4\pi n}\frac{1}{\left(1+\frac{\epsilon-1}{3}\right)}$$

relates atomic polarizability to measured dielectric constant

$$\boxed{\alpha = \frac{3}{4\pi n}\left(\frac{\epsilon-1}{\epsilon+2}\right)}$$

Clausius Mossotti or Lorentz-Lorenz equation

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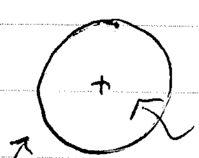
simple model for α



uniform $\rho = \frac{-q}{\frac{4}{3}\pi a^3}$

atomic radius a

field inside is $E(r) = \frac{4\pi\rho}{3} r\,\hat{r}$

~~polarization~~

In external field E_0, net forces balance $\Rightarrow q E_0 = q\frac{4\pi\rho}{3}d$

$$p = qd = \frac{3}{4\pi\rho}qE_0 = \frac{3\left(\frac{4}{3}\pi a^3\right)}{4\pi q}qE_0$$

$$= a^3 E_0 \Rightarrow \boxed{\alpha = a^3}$$

$$\chi_e = \frac{na^3}{1-\frac{4\pi}{3}na^3}$$

if $f = n\frac{4}{3}\pi a^3$ fraction of vol that is occupied by atoms

$$\boxed{\chi_e = \frac{1}{4\pi}\frac{3f}{1-f}}$$

Linear dielectrics

bound charge is proportional to free charge

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot(\chi_e\vec{E}) = -\vec{\nabla}\cdot\left(\frac{\chi_e}{\epsilon}\vec{D}\right)$$

if χ_e (and hence ϵ) is spatially constant, then

$$\rho_b = -\frac{\chi_e}{\epsilon}\vec{\nabla}\cdot\vec{D} = -\frac{\chi_e}{\epsilon}4\pi\rho$$

$$\boxed{\rho_b = -\frac{4\pi\chi_e}{1+4\pi\chi_e}\rho}$$

when free charge $\rho = 0$, then $\rho_b = 0$

$$\rho_{total} = \rho + \rho_b = \rho\left[1 - \frac{4\pi\chi_e}{1+4\pi\chi_e}\right] = \frac{\rho}{1+4\pi\chi_e} = \boxed{\frac{\rho}{\epsilon} = \rho_{total}}$$

bound charge "screens" the free charge so the total charge is reduced compared to the free charge.

For <u>linear dielectrics</u>

<u>Statics</u>

$\vec{\nabla} \cdot \vec{D} = 4\pi\rho$
$\vec{\nabla} \times \vec{E} = 0$

$\vec{D} = \epsilon \vec{E} \quad \Rightarrow \quad \vec{\nabla} \cdot (\epsilon \vec{E}) = 4\pi\rho$

if ϵ is constant in space then $\epsilon \vec{\nabla} \cdot \vec{E} = 4\pi\rho$

$$\left.\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 4\pi\rho/\epsilon = 4\pi\rho_{tot} \\ \vec{\nabla} \times \vec{E} &= 0 \end{aligned}\right\} \text{ look just like ordinary electrostatics but with } \rho \to \rho/\epsilon$$

Alternatively, could write $\vec{E} = \vec{D}/\epsilon$

$\Rightarrow \vec{\nabla} \times (\vec{D}/\epsilon) = 0$

$\Rightarrow \vec{\nabla} \times \vec{D} = 0$ when ϵ constant in space

$$\left.\begin{aligned} \vec{\nabla} \cdot \vec{D} &= 4\pi\rho \\ \vec{\nabla} \times \vec{D} &= 0 \end{aligned}\right\} \text{ looks just like ordinary electrostatics, but with } \vec{E} \to \vec{D}$$

Complication arises at <u>interface</u> between dielectrics (or between dielectric and vacuum). At interface, ϵ is <u>not</u> constant $\Rightarrow \vec{\nabla} \times \vec{D} \neq 0$.

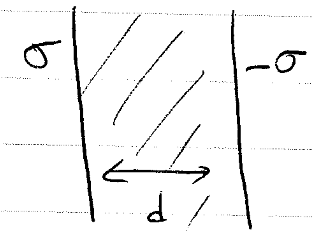
What we can do is to solve for $\vec{E}$ or $\vec{D}$ inside each dielectric separately, and then use the <u>boundary conditions</u>

$$\hat{n} \cdot (\vec{D}^{above} - \vec{D}^{below}) = 4\pi\sigma$$
$$\hat{t} \cdot (\vec{E}^{above} - \vec{E}^{below}) = 0$$

to match solutions across the interfaces.

A similar story holds for linear magnetic materials

<u>Simple example</u>: parallel plate capacitor filled with a dielectric



σ free charge

What is E between plates?

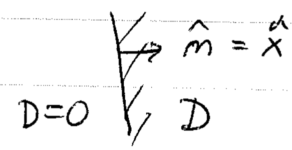
we know $\vec{E} = \vec{D} = 0$ outside plates

Between plates $\vec{\nabla} \cdot \vec{D} = 0$ as $\rho = 0$

$\vec{D} = D(x)\,\hat{x} \quad \Rightarrow \quad \frac{\partial D}{\partial x} = 0 \Rightarrow D$ is constant

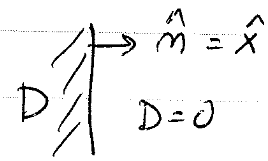
<u>Boundary conditions</u>:

left side plate



$$\hat{x} \cdot (\vec{D}^{above} - \vec{D}^{below}) = D = 4\pi\sigma$$

right side plate



$$\hat{x} \cdot (\vec{D}^{above} - \vec{D}^{below}) = -D = 4\pi(-\sigma)$$

$$D = 4\pi\sigma \quad \text{as before}$$

$$\Rightarrow \quad \vec{D} = 4\pi\sigma\,\hat{x}$$

$$\boxed{\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{4\pi\sigma}{\epsilon}\,\hat{x}}$$

electric field reduced by factor $1/\epsilon$ as compared to capacitor with vacuum between plates

see Jackson section 4.4 for more interesting examples
– dielectric sphere in uniform applied $\vec{E}$

see Jackson section $\binom{5.11}{5.12}$ for an interesting magnetic b.c. problem