Combine results to get

$$\int_V d^3r \; \vec{j}\cdot\vec{E} = -\frac{1}{4\pi}\int_V d^3r \left[\frac{1}{2}\frac{\partial B^2}{\partial t} + \frac{1}{2}\frac{\partial E^2}{\partial t} + c\,\vec{\nabla}\cdot(\vec{E}\times\vec{B}) \right]$$

define
$$\boxed{\begin{aligned} u &= \frac{1}{8\pi}(E^2+B^2) \\ \vec{S} &= \frac{c}{4\pi}\vec{E}\times\vec{B} \end{aligned}}$$

electromagnetic energy density

Poynting vector - energy current

then

$$\frac{d\mathcal{E}_{mech}}{dt} = \int_V d^3r \,\vec{j}\cdot\vec{E} = -\int_V d^3r \left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} \right]$$

If we define $\mathcal{E}_{EM}$, the electromagnetic energy of the volume V, as

$$\mathcal{E}_{EM} = \int_V d^3r \; u$$

then

$$\frac{d}{dt}\left(\mathcal{E}_{mech} + \mathcal{E}_{EM}\right) = -\oint_S da \;\hat{n}\cdot\vec{S}$$

~~or if we write $\frac{\partial u_{mech}}{\partial t} = \vec{j}\cdot\vec{E}$ as the time rate of change of mechanical~~

or we can write in differential form

$$\vec{j}\cdot\vec{E} + \frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0$$

rate of change of mechanical energy per unit volume

local energy conservation law if interpret $\vec{S}$ as energy current and u as EM energy density

$$\frac{d}{dt}\left(\mathcal{E}_{mech} + \mathcal{E}_{EM}\right) = -\oint_S da\ \hat{n}\cdot\vec{S}$$

total energy in V can decrease only if electromagnetic energy is being transported through the surface S by the EM energy current $\vec{S}$.
assumes the charged particles do not leave the volume V.

Under certain conditions, we can derive a similar conservation law for the macroscopic maxwell eqns.

Consider that $\vec{j}$ is current of the free (charged) particles.

Then repeating the above steps:

$$\int_V d^3r\ \vec{j}\cdot\vec{E} = \frac{c}{4\pi}\int d^3r\ \vec{E}\cdot\left[\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t}\right]$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$
$$= -\frac{1}{c}H\cdot\frac{\partial\vec{B}}{\partial t} - E\cdot\vec{\nabla}\times\vec{H}$$

so

$$\int_V d^3r\ \vec{j}\cdot\vec{E} = \frac{-1}{4\pi}\int_V d^3r\left[c\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right]$$

If the medium is linear, and we have quasistatic conditions, so that

$$\vec{D}(t) \simeq \epsilon\vec{E}(t)$$
$$\vec{H}(t) \simeq \frac{1}{\mu}\vec{B}(t)$$

then we can write

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \epsilon\,\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\epsilon\frac{\partial E^2}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D})$$

$$\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{\mu}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{2\mu}\frac{\partial B^2}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{B}\cdot\vec{H})$$

** Note: in general, as we will soon see, above conditions are not satisfied. ϵ will depend on frequency ω and $\vec{D}(t)$ and $\vec{E}(t)$ are non locally related in time. $\vec{D}(t) = \int_{-\infty}^{t} dt'\, \epsilon(t-t')\,\vec{E}(t')$. Only at low frequencies, ie in quasistatic case, can we write $\vec{D}(t) \simeq \epsilon(\omega=0)\,\vec{E}(t)$.

Assuming the above conditions are met, then

$$\int_V d^3r\, \vec{j}\cdot\vec{E} + \int_V d^3r\, \frac{\partial u}{\partial t} = -\oint_S da\, \hat{n}\cdot\vec{S}$$

where
$$\begin{cases} u = \dfrac{1}{8\pi}\left[\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right] \\[2ex] \vec{S} = \dfrac{c}{4\pi}\left[\vec{E}\times\vec{H}\right] \end{cases}$$

$\Rightarrow$ electromagnetic energy in dielectric + magnetic materials under quasi static conditions is

$$\int_V d^3r \left[\underbrace{\frac{1}{8\pi}\vec{E}\cdot\vec{D}}_{\text{electrostatic energy}} + \underbrace{\frac{1}{8\pi}\vec{B}\cdot\vec{H}}_{\text{magnetostatic energy}}\right]$$

Statics

Electrostatic Energy

Returning to microscopic fields and charges

$$\mathcal{E} = \frac{1}{8\pi}\int_V d^3r\, E^2 \qquad \text{use } \vec{E} = -\vec{\nabla}\phi$$

$$= \frac{-1}{8\pi}\int_V d^3r\, (\vec{\nabla}\phi)\cdot\vec{E} \qquad \text{use } \vec{\nabla}\cdot(\phi\vec{E}) = \phi\,\vec{\nabla}\cdot\vec{E} + (\vec{\nabla}\phi)\cdot\vec{E}$$

$$= \frac{-1}{8\pi}\int_V d^3r\left[\vec{\nabla}\cdot(\phi\vec{E}) - \phi\,\vec{\nabla}\cdot\vec{E}\right] \qquad \text{use } \vec{\nabla}\cdot E = 4\pi\rho$$

$$= \frac{1}{2}\int_V d^3r\, \rho\phi - \frac{1}{8\pi}\oint_S da\, \hat{n}\cdot\phi\vec{E} \qquad \text{by Gauss Theorem}$$

If let V be all space, $S\to\infty$, then $\phi \sim \frac{1}{r}$, $E \sim \frac{1}{r^2}$
surface integral $\sim \frac{R^2}{R^3} \to 0$ as $R\to\infty$.

$$\boxed{\mathcal{E} = \frac{1}{2}\int d^3r\, \rho\phi}$$

can also use $\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$ to write

$$\boxed{\mathcal{E} = \frac{1}{2}\int d^3r\, d^3r' \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}}$$

charge-charge interaction

Magnetostatic Energy

microscopic fields and currents

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r\ B^2 \qquad \text{use } \vec{B} = \vec{\nabla}\times\vec{A}$$

$$= \frac{1}{8\pi}\int d^3r\ \vec{B}\cdot\vec{\nabla}\times\vec{A} \qquad \text{use } \vec{\nabla}\cdot(\vec{B}\times\vec{A}) = \vec{A}\cdot(\vec{\nabla}\times\vec{B}) - \vec{B}\cdot(\vec{\nabla}\times\vec{A})$$

$$= \frac{1}{8\pi}\int d^3r\left[\vec{A}\cdot\vec{\nabla}\times\vec{B} - \vec{\nabla}\cdot(\vec{B}\times\vec{A})\right] \qquad \text{use } \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}$$

$$= \frac{1}{2c}\int d^3r\ \vec{j}\cdot\vec{A} - \frac{1}{8\pi}\oint_S da\ \hat{n}\cdot(\vec{B}\times\vec{A})$$

as take V to fill all space, $S\to\infty$, surface term vanishes

$$\boxed{\mathcal{E} = \frac{1}{2c}\int d^3r\ \vec{j}\cdot\vec{A}}$$

In Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$, $\vec{A}(\vec{r}) = \int d^3r'\ \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$

In any other gauge we have $\vec{A}' = \vec{A} + \vec{\nabla}\chi$ for some scalar χ. So we can always write

$$\vec{A}(\vec{r}) = \int d^3r'\ \frac{\vec{j}(\vec{r}')}{c\,|\vec{r}-\vec{r}'|} + \vec{\nabla}\chi$$

regardless of the choice of gauge, where χ is then determined so $\vec{A}$ satisfies the desired gauge condition

$$\mathcal{E} = \frac{1}{2c}\int d^3r\, d^3r' \,\frac{\vec{j}(\vec{r})\cdot\vec{j}(\vec{r}')}{c\,|\vec{r}-\vec{r}'|} + \frac{1}{2c^2}\int d^3r\, \vec{j}\cdot\vec{\nabla}\chi$$

2nd term is $\int d^3r\, \vec{j}\cdot\nabla\chi = \int d^3r \left[\nabla\cdot(\vec{j}\chi) - \chi\,\vec{\nabla}\cdot\vec{j}\right]$

$$= \oint_S da\, \hat{n}\cdot\vec{j}\chi - \int d^3r\, \chi\,\vec{\nabla}\cdot\vec{j}$$

vanishes as $S \to \infty$

vanishes in magnetostatics where $\vec{\nabla}\cdot\vec{j} = 0$

So

$$\boxed{\mathcal{E} = \frac{1}{2c^2}\int d^3r\, d^3r' \,\frac{\vec{j}(\vec{r})\cdot\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}}$$

current-current interaction

Momentum Conservation

For charges q_i at positions $\vec{r}_i$ with velocities $\dot{\vec{r}}_i$

$$\frac{d\vec{P}^{\,mech}}{dt} = \sum_i \vec{F}_i = \sum_i q_i\left(\vec{E}(\vec{r}_i) + \frac{1}{c}\vec{v}_i \times \vec{B}(\vec{r}_i)\right)$$

"mechanical" momentum of the charges

force on charge i

$$= \int_V d^3r \left[\rho\vec{E} + \frac{1}{c}\vec{J}\times\vec{B}\right]$$

$$\rho E + \frac{1}{c}\vec{J}\times\vec{B} = \frac{1}{4\pi}\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + \left(\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}\right]$$

Now $\frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{1}{c}\left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right) + \frac{1}{c}\left(\vec{E}\times\frac{\partial\vec{B}}{\partial t}\right)$ use $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$

$$= \frac{1}{c}\left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right) - \vec{E}\times(\vec{\nabla}\times\vec{E})$$

So $-\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\times\vec{B} = -\vec{E}\times(\vec{\nabla}\times\vec{E}) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$

Therefore

$$\rho E + \frac{1}{c}\vec{J}\times\vec{B} = \frac{1}{4\pi}\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + \vec{B}\underbrace{(\vec{\nabla}\cdot\vec{B})}_{=0} - \vec{B}\times(\vec{\nabla}\times\vec{B}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})\right]$$

Define electromagnetic momentum density

$$\boxed{\vec{\Pi} = \frac{1}{4\pi c}\vec{E}\times\vec{B} = \frac{1}{c^2}\vec{S}}$$ ($\vec{S}$ is Poynting vector)

then

$$\frac{d\vec{P}^{\,mech}}{dt} + \frac{d}{dt}\int_V d^3r\,\vec{\Pi} = \frac{1}{4\pi}\int_V d^3r\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) + \vec{B}(\vec{\nabla}\cdot\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{B})\right]$$

want to rewrite as a surface integral

ith component of integrand on right hand side is ($\vec{E}$ part only)
(sum over repeated indicies)

$$E_i \partial_j E_j - \varepsilon_{ijk} E_j \varepsilon_{klm} \partial_l E_m$$

$$= E_i \partial_j E_j - \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) E_j \partial_l E_m$$

$$= E_i \partial_j E_j - E_j \partial_i E_j + E_j \partial_j E_i$$

$$= \partial_j \left(E_i E_j - \tfrac{1}{2}\delta_{ij} E^2\right)$$

Define <u>Maxwell's stress tensor</u>

$$\boxed{T_{ij} = \frac{1}{4\pi}\left[E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}\left(E^2 + B^2\right)\right]}$$

(note $T_{ij} = T_{ji}$ symmetric tensor

Then

$$\frac{d}{dt} p_i^{mech} + \frac{d}{dt}\int_V d^3r\, \Pi_i = \int_V d^3r\, \partial_j T_{ij}$$

$$\left(\partial_j T_{ij} = \frac{\partial T_{ij}}{\partial x_j}\right)$$

$$= \oint_S da\, T_{ij} \circ \hat{n}_j$$

$$\frac{d}{dt}\vec{p}^{\,mech} + \frac{d}{dt}\int_V d^3r\, \vec{\Pi} = \oint_S da\, \overleftrightarrow{T} \circ \hat{n}$$

- T_{ij} gives the flow of the i^{th} component of electromagnetic field momentum through an element of surface area $\perp$ ~~to~~ to direction $\hat{e}_j$

~~For static situations where~~ $\cancel{\frac{\partial \vec{\Pi}}{\partial t} = 0}$, $\cancel{\frac{d\vec{p}^{\,mech}}{dt} = \vec{F}^{tot} = \oint_S da\, \overleftrightarrow{T} \circ \hat{n}}$
~~gives electromagnetic force on the surface S~~

Note: $\frac{d\vec{P}^{mech}}{dt}$ is ~~also~~ equal to the total electromagnetic force on the volume V.

Hence we can write

$$\vec{F}_{EM} = \oint_S da\, \overleftrightarrow{T}\cdot\hat{n} - \frac{d}{dt}\int_V d^3r\, \vec{\Pi}$$

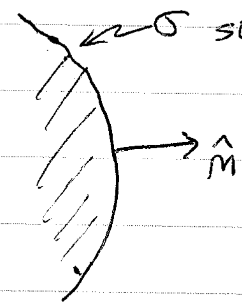
for <u>static</u> situations, the 2nd term vanishes and

$$\vec{F}_{EM} = \oint_S da\, \overleftrightarrow{T}\cdot\hat{n}$$

$\overleftrightarrow{T}_{ij}$ is i^{th} component of static force on unit area with normal $\hat{e}_j$

this is origin of the term "stress" tensor.
$\overleftrightarrow{T}$ is like the stress tensor of an elastic medium,
T_{xx}, T_{yy}, T_{zz} are like pressure.
Off diagonal elements are like shear stresses

Force on a conductor surface



σ surface charge on conductor

net force on surface per unit area is

$$\vec{f} = \overleftrightarrow{T}^{above} \cdot \hat{n} - \underbrace{\overleftrightarrow{T}^{below} \cdot \hat{n}}$$

$= 0$ as $\vec{E} = 0$ inside conductor

$$\vec{f} = \frac{1}{4\pi}\left[\vec{E}(\vec{E}\cdot\hat{n}) - \frac{1}{2}\hat{n}E^2\right]$$

for conducting surface

$$\hat{n} \cdot \vec{E}^{above} = 4\pi\sigma \quad (\text{since } \vec{E} \text{ below} = 0)$$

and tangential component $\vec{E} = 0$

$$\Rightarrow \vec{E} = 4\pi\sigma\,\hat{n}$$

So $$\vec{f} = \frac{1}{4\pi}\left[(4\pi\sigma\,\hat{n})(4\pi\sigma) - \frac{1}{2}\hat{n}(4\pi\sigma)^2\right]$$

$$\boxed{\cancel{\vec{f} = 2\pi\sigma^2\hat{n}}}$$

$$\vec{f} = \frac{\hat{n}}{4\pi}\left[(4\pi\sigma)^2 - \frac{1}{2}(4\pi\sigma)^2\right] = 2\pi\sigma^2\hat{n}$$

force per unit area: $$\boxed{\vec{f} = 2\pi\sigma^2\hat{n} = \frac{1}{2}\sigma\vec{E}}$$

$\vec{f} = \sigma\vec{E}_{ave}$

where $\vec{E}_{ave} = \frac{1}{2}(\vec{E}_{above} + \vec{E}_{below})$

is average field at surface averaging over above + below

Note factor 1/2. Naively one might have thought $\vec{f} = \sigma\vec{E}$. But need to exclude self field of charge on surface from acting on itself. See also Jackson pg 42 for another approach.