Capacitance

Consider a set of conductors with potential \( \phi(\vec{r}) = V_i \) fixed on conductor \( i \).

(also need condition on \( \phi(\vec{r}) \to \infty \) if system is not enclosed)

From uniqueness theorem we know that specifying the \( V_i \) on each conductor is enough to determine the potential \( \phi(\vec{r}) \) everywhere. We can write this potential in the following form.

Let \( \phi^{(i)}(\vec{r}) \) be the solution to the boundary value problem
\[
\nabla^2 \phi^{(i)}(\vec{r}) = 0 \quad \text{and} \quad \phi^{(i)}(\vec{r}) = \begin{cases} V_i & \text{if } \vec{r} \text{ on surface of conductor } (i) \\ 0 & \text{if } \vec{r} \text{ on surface of any other } \text{conductor } (j), \ j \neq i \end{cases}
\]

Then by superposition
\[
\phi(\vec{r}) = \sum_i V_i \phi^{(i)}(\vec{r})
\]

is solution to the problem \( \nabla^2 \phi = 0 \) and \( \phi(\vec{r}) = V_i \) for \( \vec{r} \) on surface of conductor \( (i) \).

The surface charge density at \( \vec{r} \) on surface of conductor \( (i) \) is
\[
\sigma^{(i)}(\vec{r}) = \frac{1}{4\pi} \frac{\partial \phi}{\partial n} = -\frac{1}{4\pi} \sum_i V_i \frac{\partial \phi^{(i)}}{\partial n}
\]

where \( \frac{\partial \phi}{\partial n} = (\nabla \phi \cdot \hat{n}) \) is the derivative normal to the surface at point \( \vec{r} \).
The total charge on conductor (i) is

\[ Q_i = \int_{s_i} d\alpha \sigma^{(i)} (\vec{r}) = -\frac{1}{4\pi} \sum_j V_j \int_{s_i} d\alpha \frac{\partial \phi^{(j)}}{\partial m} \]

↑

Surface of conductor (i)

Define \[ C_{ij} = -\frac{1}{4\pi} \int_{s_i} d\alpha \frac{\partial \phi^{(j)}}{\partial m} \]

the \( C_{ij} \) depend only on the geometry of the conductors

Then we have

\[ Q_i = \sum_j C_{ij} V_j \]

↑

\( C_{ij} \) is the capacitance matrix

The charge on conductor (i) is a linear function of the potentials \( V_j \) on the conductors (j)

Since we know that specifying the \( Q_i \) that is on each conductor will uniquely determine \( \phi (\vec{r}) \) and hence the potential \( V_i \) on each conductor, the capacitance matrix is invertible

\[ V_i = \sum_j [C^{-1}]_{ij} Q_j \]

The electrostatic energy of the conductors is then

\[ E = \frac{1}{2} \int d^3r \rho \phi = \frac{1}{\varepsilon} \sum_i Q_i V_i = \frac{1}{2} \sum_{ij} C_{ij} V_i V_j = \frac{1}{2} \sum_{ij} C_{ij} Q_i Q_j \]
Capacitors define capacitance of two conductors by

\[ C = \frac{Q}{V_1 - V_2} \]

when conductor (1) has charge \( Q \), conductor (2) has charge \( -Q \)

\( V_1 - V_2 \) is potential difference between the two conductors.

We can determine \( C \) in terms of the elements of the matrix \( C_{ij} \):

\[
Q = C_{11} V_1 + C_{12} V_2 \\
-\mathbf{Q} = C_{21} V_1 + C_{22} V_2 \quad \Rightarrow \quad V_2 = -\left( \frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right) V_1
\]

\[ \mathbf{Q} = \begin{bmatrix} C_{11} - C_{12} \left( \frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right) \\ V_1 - V_2 = \begin{bmatrix} 1 + \left( \frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right) \end{bmatrix} V_1
\]

\[ C = \frac{Q}{V_1 - V_2} = \frac{C_{11} - C_{12} \left( \frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right)}{1 + \left( \frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right)}
\]

\[ C = \frac{C_{11} C_{22} - C_{12} C_{21}}{C_{11} + C_{12} + C_{21} + C_{22}}
\]

Capacitance can also be defined when the space between the conductors is filled with a dielectric \( \varepsilon \). In this case, if \( Q_1 \) is the free charge, then \( Q_1 / \varepsilon \) is the effective total charge to use in computing \( C \).
\[ \frac{Q_i}{\varepsilon} = \sum_j C_{ij}^{(0)} V_j \]
\[ A_i = \sum_j \varepsilon C_{ij}^{(0)} V_j \]
\[ = \sum_j C_{ij} V_j \quad \text{where} \quad C_{ij} = \varepsilon C_{ij}^{(0)} \]

the capacitance is increased by a factor the dielectric constant $\varepsilon$. 

where $C_{ij}^{(0)}$ are capacitances appropriate to a vacuum between the conductors.
Consider a set of current carrying loops $C_i$ with currents $I_i$.

In Coulomb gauge, we can write the magnetic vector potential $\mathbf{A}$ from these current loops as

$$
\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int d^3 \mathbf{r}' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{c} \sum_i I_i \oint_{C_i} \frac{d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|}
$$

Integrate over loop $C_i$, integration variable is $\mathbf{r}'$.

The magnetic flux through loop $C_i$ is

$$
\Phi_i = \oint_{S_i} \mathbf{A} \cdot d\mathbf{S} = \oint_{S_i} \mathbf{A} \cdot \nabla \times \mathbf{A} = \oint_{C_i} d\mathbf{l} \cdot \mathbf{A}
$$

surface bounded by loop $C_i$

$$
\Phi_i = \sum_j \frac{I_j}{c} \oint_{C_i} \oint_{C_j} \frac{d\mathbf{l}_i \cdot d\mathbf{l}_j}{|\mathbf{r} - \mathbf{r}'|}
$$

pure geometrical quantity

$$
\Phi_i = c \sum_j M_{ij} I_j
$$

where

$$
M_{ij} = \oint_{C_i} \oint_{C_j} \frac{d\mathbf{l}_i \cdot d\mathbf{l}_j}{|\mathbf{r} - \mathbf{r}'|}
$$

is the mutual inductance of loops $C_i$ and $C_j$. $M_{ii} = M_{ii}$. 

In loop notation,

$$
\Phi_i = c \sum_j M_{ij} I_j
$$

The magnetic flux through loop $C_i$ is

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\Phi_i = c \sum_j M_{ij} I_j
$$

where

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M_{ij} = \oint_{C_i} \oint_{C_j} \frac{d\mathbf{l}_i \cdot d\mathbf{l}_j}{|\mathbf{r} - \mathbf{r}'|}
$$

is the mutual inductance of loops $C_i$ and $C_j$. $M_{ii} = M_{ii}$. 

In loop notation,

$$
\Phi_i = c \sum_j M_{ij} I_j
$$
\[ L_i = M_{ii} \text{ is self-inductance of loop (i)} \]

The sign convention in the above is that \( \Phi_i \) is computed in direction given by right-hand rule, according to the direction taken for current in loop (i).

\[ \Phi_i \]

\[ I_i \]

**Magnetic static energy**

\[ E = \frac{1}{2c} \int d^3r \, \vec{j} \cdot \vec{A} = \frac{1}{2c} \sum_i \oint d\vec{l} \cdot \vec{A} \cdot I_i \]

\[ = \frac{1}{2c} \sum_i \Phi_i \cdot I_i \]

\[ E = \frac{1}{2} \sum_{i,j} M_{ij} I_i I_j \]
Electromagnetic Waves in a Vacuum

No sources $\vec{f} = 0, \quad \rho = 0$

1) $\vec{\nabla} \cdot \vec{E} = 0$
2) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{c \partial t}$
3) $\vec{\nabla} \cdot \vec{B} = 0$
4) $\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$

$\nabla \times (\nabla \times \vec{E}) = \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$ (by (1))

$- \nabla^2 \vec{E} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$

$\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0$

Similarly

Note: In MKS units, above wave equation looks like

$\nabla^2 \vec{E} - \frac{1}{\varepsilon_0 \mu_0} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$

It was noticed that the speed of electromagnetic wave,

$\frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \text{ m/s}$

was the same as the speed of light! This observation was a key element in showing that light was in fact electromagnetic waves.
Harmonic Plane Waves

\[ \vec{E}(\vec{r}, t) = \text{Re} \left[ \frac{\vec{E}_k e^{i(\vec{k} \cdot \vec{r} - \omega t)}}{\text{Re} \left[ \frac{\vec{B}_k e^{i(\vec{k} \cdot \vec{r} - \omega t)}}{\vec{k}} \right]} \right] \]

- \( \vec{k} \) is wave vector
- \( \omega \) is angular frequency
- \( \nu = \frac{\omega}{2\pi} \) is frequency
- \( T = \frac{1}{\nu} \) is period
- \( \lambda = \frac{2\pi}{|\vec{k}|} \) is wavelength

\[ \frac{|\vec{E}_k|}{|\vec{B}_k|} \text{ is amplitude} \]

\[ \vec{E}(\vec{r} + \lambda \hat{k}, t) = \vec{E}(\vec{r}, t) \] periodic in space with period \( \lambda \)

\[ \vec{E}(\vec{r}, t + T) = \vec{E}(\vec{r}, t) \] periodic in time with period \( T \)

"Plane wave" \( \Rightarrow \vec{E}(\vec{r}, t) \) is constant on space on planes with normal \( \hat{\vec{n}} \parallel \hat{\vec{k}} \).

Properties of EM Plane Waves

\[ \vec{\nabla} \cdot \vec{E} = 0 \Rightarrow \text{Re} \left[ \vec{E}_k \cdot \vec{\nabla} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right] \]

\[ = \text{Re} \left[ i \vec{E}_k \cdot \vec{k} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right] = 0 \]

\[ \Rightarrow \vec{E}_k \cdot \vec{k} = 0 \]

Amplitude is orthogonal to \( \hat{\vec{k}} \)

\[ \vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B}_k \cdot \vec{k} = 0 \]

Amplitude orthogonal to \( \hat{\vec{k}} \)
\[ \nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \]

\[ \Rightarrow \text{Re} \left[ \nabla \times \vec{B}_h e^{i(k \cdot r - wt)} \right] = \text{Re} \left[ \frac{1}{c} \vec{E}_k \frac{\partial}{\partial t} e^{i(k \cdot r - wt)} \right] \]

\[ \Rightarrow \text{Re} \left[ -\vec{B}_k \times \vec{E}_k e^{i(k \cdot r - wt)} \right] = \text{Re} \left[ -i \frac{\omega}{c} \vec{E}_k e^{i(k \cdot r - wt)} \right] \]

\[ \Rightarrow \text{Re} \left[ i \vec{k} \times \vec{B}_h e^{i(k \cdot r - wt)} \right] = \text{Re} \left[ -i \frac{\omega}{c} \vec{E}_k e^{i(k \cdot r - wt)} \right] \]

\[ \Rightarrow \vec{k} \times \vec{B}_h = -\frac{\omega}{c} \vec{E}_k \]

\[ \vec{k} \times \vec{k} \times \vec{B}_h = -k^2 \vec{B}_h = -\frac{\omega}{c} \vec{k} \times \vec{E}_k \]

\[ \vec{B}_h = \frac{\omega}{ck^2} \vec{k} \times \vec{E}_k \]

Finally,

\[ \nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \]

\[ \Rightarrow \text{Re} \left[ \vec{E}_k \nabla^2 e^{i(k \cdot r - wt)} - \frac{\vec{E}_k}{c^2} \frac{\partial^2}{\partial t^2} e^{i(k \cdot r - wt)} \right] = 0 \]

\[ \Rightarrow \text{Re} \left[ \vec{E}_k (-k^2) e^{i(k \cdot r - wt)} + \frac{\omega^2}{c^2} \vec{E}_k e^{i(k \cdot r - wt)} \right] = 0 \]

\[ \Rightarrow k^2 = \frac{\omega^2}{c^2} \]

\[ \omega = \pm kc \]

\[ \text{dispersion relation} \]

consistent with above

\[ \vec{B}_h = \hat{k} \times \vec{E}_k \]

\[ \hat{k} = \frac{\vec{E}_k}{|\vec{E}_k|} \]

\[ \Rightarrow |\vec{k}| = |\vec{E}_k| \]
\[ \vec{E}_k \perp \vec{k} \quad \implies \quad \text{"transverse" polarization} \]
\[ \vec{B}_k \perp \vec{k} \]
\[ \vec{B}_k = \vec{k} \times \vec{E}_k \]
\[ \omega^2 = c^2 k^2 \]

\[ |\vec{B}_k| = 1 \text{c}|\vec{E}_k| \Rightarrow \text{ Lorentz force from plane EM wave on charge q is } \]
\[ \vec{F} = \frac{q}{c} \left( \vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right) \]

magnetic force = smaller factor \((\frac{\omega}{c})\) as compared to electric force - can usually be ignored.

Most general solution is a linear superposition of the above plane wave:

\[ \vec{E}(\vec{r}, t) = \int \frac{d^3k}{(2\pi)^3} \vec{E}_k e^{i(k \cdot \vec{r} - \omega t)} \]

Formal transform

\[ \vec{E}(\vec{r}, t) \quad \text{is real} \Rightarrow \vec{E}_k^* = \vec{E}_{-k} \]

For dispersion relation \(\omega^2 = c^2 k^2\), we can write

\[ \vec{k} \cdot \vec{r} - \omega t = \vec{k} \cdot (\vec{r} - \vec{v} t) \]

where \(\vec{v} = c \hat{k}\) is velocity of wave. If we only consider waves traveling in same direction \(\vec{k}\), the plane wave

\[ \vec{E}(\vec{r}, t) = \int \frac{d^3k}{(2\pi)^3} \vec{E}_k e^{i(k \cdot (\vec{r} - \vec{v} t))} = \vec{E}(\vec{r} - \vec{v} t, 0) \]

The general solution of wave equation always has this property:

\[ \vec{E}(\vec{r}, t) = \vec{E}(\vec{r} - \vec{v} t, 0) \]

If know \(\vec{E}\) at \(t=0\), then...