Energy & momentum in EM wave

$$\vec{E} = Re\left[\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \vec{E}_k \cos(\vec{k}\cdot\vec{r}-\omega t)$$
$$\vec{B} = Re\left[\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \hat{k}\times\vec{E}_k \cos(\vec{k}\cdot\vec{r}-\omega t)$$

for real $\vec{E}_k$

energy density $\quad u = \frac{1}{8\pi}\left(E^2 + B^2\right)$

$$= \frac{1}{8\pi}\left[E_k^2 + E_k^2\right]\cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$= \frac{1}{4\pi} E_k^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

Poynting vector

energy current $\quad \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$

$$= \frac{c}{4\pi}\left[\vec{E}_k \times (\hat{k}\times\vec{E}_k)\right]\cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$= \frac{c}{4\pi}\hat{k} E_k^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$\vec{S} = c\, u\, \hat{k}$$

momentum density $\quad \vec{\Pi} = \frac{1}{c^2}\vec{S} = \frac{u}{c}\hat{k}$

$$u = c|\vec{\Pi}|$$ — energy momentum relation of photons!

For visible light $\lambda \sim 5\times10^{-7}$ m ~ 5000 Å

$$T = \frac{\lambda}{c} = 1.6\times10^{-15} \text{ sec}$$

most classical measurements on microscopic scales $\quad t \gg T, \quad \ell \gg \lambda$

measure average quantities

$$\langle u \rangle = \frac{1}{T}\int_0^T dt\ u = \frac{1}{8\pi} E_k^2 \qquad \text{as } \langle \cos^2\theta \rangle = 1/2$$

$$\langle \vec{S} \rangle = c \langle u \rangle \hat{k}$$

$$\langle \vec{\Pi} \rangle = \frac{1}{c}\langle u \rangle \hat{k}$$

intensity = average power per area transported by wave through surface with normal $\hat{n}$

$$I = \langle \vec{S} \rangle \cdot \hat{n}$$

Electromagnetic waves in matter

Macroscopic Maxwell equations with no sources ("free" charge and current vanishes)

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

linear materials

$$\vec{B} = \mu\vec{H}$$
$$\vec{D} = \epsilon\vec{E}$$

if μ and ϵ were simply constants then the above would become

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{\mu\epsilon}{c}\frac{\partial \vec{E}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

Then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2 E = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$
$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mu\epsilon}{c}\frac{\partial E}{\partial t}\right)$$
$$= -\frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2}$$

wave equation with wave speed $\frac{c}{\sqrt{\mu\epsilon}} < c$

This would be very much as for waves in a vacuum, except for the following minor changes:

$$\omega^2 = \frac{c^2 k^2}{\mu \epsilon}$$ dispersion relation changed by constant factor

$$\vec{E}_k \perp \vec{k}$$
$$\vec{B}_k \perp \vec{k}$$

$$i \vec{k} \times \vec{E}_k = \frac{i\omega}{c} \vec{B}_k$$

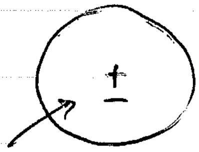
$$\frac{c|\vec{k}|}{\omega} \hat{k} \times \vec{E}_k = \vec{B}_k$$

$$\Rightarrow \sqrt{\mu\epsilon}\, \hat{k} \times \vec{E}_k = \vec{B}_k \qquad |\vec{B}_k| > |\vec{E}_k|$$

wave speed $v = \frac{c}{\sqrt{\mu\epsilon}} < c$

In general however, things are <u>much more complicated</u> because ϵ cannot be viewed as a constant when considering time varying behavior!

Time dependent polarizeability of an atom



electron cloud

If displace center of electron cloud by a distance $\vec{r}$, there is a restoring force $\vec{F}_{rest} = -\dfrac{e^2 \vec{r}}{4\pi R^3} \equiv -m\omega_0^2 \vec{r}$

electron mass ↗ ↑ resonant frequency

Also, in general there will be a damping force

$$\vec{F}_{damp} = -m\gamma \frac{d\vec{r}}{dt}$$

due to transfer of energy from atom to other degrees of freedom.

In an external electric field $\vec{E}(t)$, the equation of motion for electron cloud is

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{tot} = -e\vec{E}(t) - m\omega_0^2\vec{r} - m\gamma\frac{d\vec{r}}{dt}$$

$$\ddot{\vec{r}} + \gamma\dot{\vec{r}} + \omega_0^2\vec{r} = -\frac{e\vec{E}(t)}{m}$$

assume $\vec{E}$ is spatially constant over atomic distances

For harmonic oscillation $\vec{E}(t) = \vec{E}_0 e^{-i\omega t}$

Assume solution $\vec{r}(t) = \vec{r}_0 e^{-i\omega t}$

(in the end, we will take the real parts)

Substitute into equation of motion

$$-\omega^2\vec{r}_0 - i\omega\gamma_0\vec{r}_0 + \omega_0^2\vec{r}_0 = -\frac{e\vec{E}_0}{m}$$

$$\vec{r}_0 = \frac{-e}{m(\omega_0^2 - \omega^2 - i\omega\gamma)}\vec{E}_0$$

polarization

$$\vec{p} = -e\vec{r} = \vec{p}_0 e^{-i\omega t}$$

$$\vec{p}_0 = \frac{e^2}{m} \frac{1}{(\omega_0^2 - \omega^2 - i\omega\gamma)} \vec{E}_0 = \alpha(\omega) E_0$$

$$\alpha(\omega) = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$ freq dependent polarizability

Since α is complex the polarization does not in general oscillate in phase with $\vec{E}$.

If $\alpha(\omega) = |\alpha| e^{i\delta}$ δ is phase of complex α

$$\vec{p}(t) = \alpha(\omega) \vec{E}(t) = |\alpha| e^{i\delta} \vec{E}_0 e^{-i\omega t} = |\alpha| \vec{E}_0 e^{-i(\omega t - \delta)}$$

phase shifted by δ

For a general electric field

$$\vec{E}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \vec{E}_\omega e^{-i\omega t}$$

$$\vec{E}_\omega^* = \vec{E}_{-\omega}$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} p_\omega e^{-i\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) \vec{E}_\omega e^{-i\omega t}$$

Substitute in $E_\omega = \int_{-\infty}^{\infty} dt' \, E(t') e^{i\omega t'}$ to get

$$\vec{p}(t) = \int_{-\infty}^{\infty} dt' \, \vec{E}(t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) e^{-i\omega(t-t')}$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} dt' \, \vec{E}(t') \tilde{\alpha}(t-t')$$

Fourier transf of $\alpha(\omega)$

$\vec{p}$ at time t is due to $\vec{E}$ at all times t'

non local in time

$\chi(t)$ is the response to $\vec{E}(t) = \delta(t)$

For our simple model

$$\tilde{\chi}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

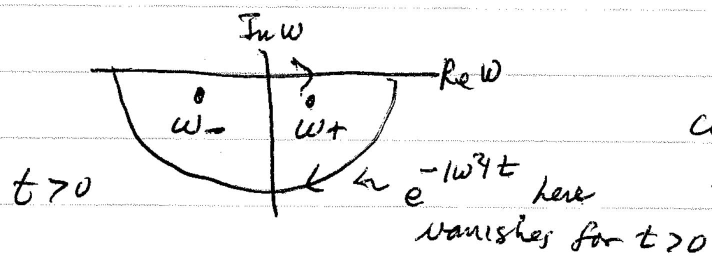
do by contour integration

$$\frac{1}{\omega^2 + i\gamma\omega - \omega_0^2} = \frac{1}{(\omega - \omega_+)(\omega - \omega_-)}$$

$$\omega_\pm = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \gamma^2/4} = -\frac{i\gamma}{2} \pm \bar{\omega}$$

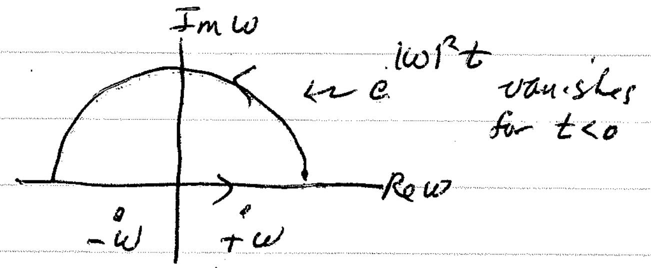
poles at $\omega_\pm$ are in lower half complex plane.

for $t > 0$, close contour in lower half plane



contour encloses poles + get contribution

for $t < 0$, close contour in upper half plane



contour encloses no poles ⇒ integral vanishes

$$\chi(t) = 0 \quad \text{for } t < 0$$

<u>causal response</u>! No polarization until electric field turns on

For $t > 0$

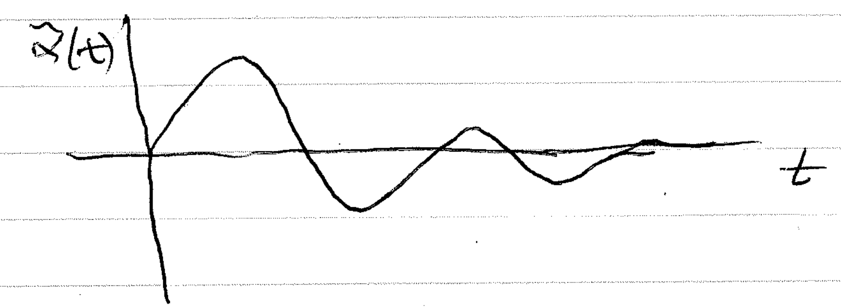
$$\tilde{\alpha}(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{e^2}{m} \frac{(-1)}{(\omega-\omega_+)(\omega-\omega_-)}$$

$$= (-2\pi i) \frac{e^2}{m} \frac{(-1)}{2\pi} \left[\frac{e^{-i\omega_+ t}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_- t}}{\omega_- - \omega_+} \right]$$

from Residue theorem

$$= \frac{i e^2}{m} \left[\frac{e^{-\gamma t/2} e^{-i\bar{\omega} t}}{2\bar{\omega}} - \frac{e^{-\gamma t/2} e^{i\bar{\omega} t}}{2\bar{\omega}} \right]$$

$$\tilde{\alpha}(t) = \begin{cases} \frac{e^2}{m} \frac{e^{-\gamma t/2}}{\bar{\omega}} \sin(\bar{\omega} t) & t > 0 \\ 0 & t < 0 \end{cases}$$



damped oscillation

Polarization density $\vec{P}_\omega = 4\pi \chi(\omega) \vec{E}_\omega$ for harmonic oscillation

$\chi(\omega) \approx n\, \alpha(\omega)$ for dilute system

atom density

can use Clausius - Mossotti correction for denser materials

$\Rightarrow \vec{D}_\omega = \varepsilon(\omega) \vec{E}_\omega$ $\qquad \varepsilon(\omega) = 1 + 4\pi\chi(\omega)$

freq dependent

→ as with $\vec{P}$ and $\vec{E}$, relation between $\vec{D}$ and $\vec{E}$ is <u>non-local</u> in time

$$\vec{D}(t) \neq \varepsilon \vec{E}(t)$$

rather

$$\vec{D}(t) = \int_{-\infty}^{\infty} dt' \, \vec{E}(t') \, \tilde{\varepsilon}(t-t')$$

↑ Fourier transf of $\varepsilon(\omega)$

Ampere's law is

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$$

becomes $$\frac{1}{\mu} \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \int_{-\infty}^{\infty} dt' \, \vec{E}(t') \frac{d}{dt} \tilde{\varepsilon}(t-t')$$

integro-differential-equation!

Maxwells equations only look simple when expressed in terms of Fourier transforms

$$\vec{E}(\vec{r},t) = \vec{E}_\omega \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$
$$\vec{B}(\vec{r},t) = \vec{B}_\omega \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$
$$\vec{D}(\vec{r},t) = \vec{D}_\omega \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$
$$\vec{H}(\vec{r},t) = \vec{H}_\omega \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Maxwell's Equ for <u>source free</u> system $\rho = \vec{j} = 0$

$$\vec{\nabla} \cdot \vec{D} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = \frac{-\partial \vec{B}}{c\,\partial t}, \quad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{c\,\partial t}$$