

assume μ is true constant - not freq dependent
dielectric response is $\vec{D}_\omega = \epsilon(\omega) \vec{E}_\omega$

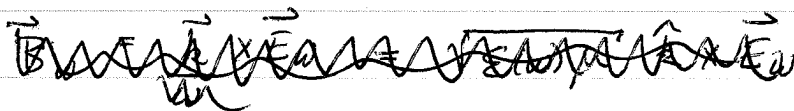
Then for the Fourier amplitudes of the fields, Maxwell's Equations become

$$\begin{aligned} 1) \quad i \vec{k} \cdot \vec{D}_\omega &= i \epsilon(\omega) \vec{k} \cdot \vec{E}_\omega = 0 & \Rightarrow \boxed{\vec{k} \perp \vec{E}_\omega} \quad (\text{unless } \epsilon(\omega) = 0) \\ 2) \quad i \vec{k} \cdot \vec{B}_\omega &= 0 & \Rightarrow \boxed{\vec{k} \perp \vec{B}_\omega} \\ 3) \quad i \vec{k} \times \vec{E}_\omega &= i \frac{\omega}{c} \vec{B}_\omega \\ 4) \quad i \vec{k} \times \vec{H}_\omega &= -i \frac{\omega}{c} \vec{D}_\omega \Rightarrow \frac{i \vec{k}}{\mu} \times \vec{B}_\omega = -\frac{i \omega}{c} \epsilon(\omega) \vec{E}_\omega \end{aligned}$$

transverse polarized

$$\begin{aligned} \vec{k} \times (3) &= i \vec{k} \times (\vec{k} \times \vec{E}_\omega) = i \frac{\omega}{c} \vec{k} \times \vec{B}_\omega \\ &\Rightarrow -i k^2 \vec{E}_\omega = -i \frac{\omega^2}{c^2} \epsilon(\omega) \mu \vec{E}_\omega \quad \text{using (4)} \end{aligned}$$

$$\boxed{k^2 = \frac{\omega^2}{c^2} \epsilon(\omega) \mu} \quad \text{dispersion relation}$$



Note: $\frac{\omega}{|k|} = \frac{c}{\sqrt{\epsilon(\omega) \mu}}$

varies with ω .

there is not a single phase velocity.

$\Rightarrow \vec{E}$ is not in general a solution of a wave equation - different frequencies travel with different speeds

Since $\epsilon(\omega)$ is complex $\epsilon(\omega) = \epsilon_1(\omega) + i\epsilon_2(\omega)$

\Rightarrow wave vector also complex For $\vec{k} = k \hat{z}$

$$k = k_1 + ik_2 = \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\epsilon_1 + i\epsilon_2}$$

$$\begin{aligned} \vec{E}(\vec{r}, t) &= \vec{E}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \vec{E}_\omega e^{i[(k_1 + ik_2)z - \omega t]} \\ &= \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)} \end{aligned}$$

k_1 determines the oscillation of the wave

k_2 determines the decay or attenuation of the wave as it propagates into the material

phase velocity $v_p = \frac{\omega}{k_1}$

index of refraction $n = \frac{c}{v_p} = \frac{ck_1}{\omega}$

group velocity $v_g = \frac{1}{\frac{dk_1}{d\omega}}$

Magnetic field: $\vec{B}_\omega = \frac{c\vec{k}}{\omega} \times \vec{E}_\omega$

for $\vec{k} = k \hat{z}$, $\vec{B}_\omega = \frac{c(k_1 + ik_2)}{\omega} \hat{z} \times \vec{E}_\omega$

if $k_1 + ik_2 = \sqrt{k_1^2 + k_2^2} e^{i\delta}$ $\delta = \arctan\left(\frac{k_2}{k_1}\right)$
 $= |k| e^{i\delta}$

$$\vec{B}_\omega = \frac{c|k|}{\omega} \hat{z} \times \vec{E}_\omega e^{i\delta}$$

\uparrow phase shift

$$\vec{B}(\vec{r}, t) = \frac{c|k|}{\omega} (\hat{z} \times \vec{E}_\omega) e^{-k_2 z} e^{i(k_1 z - \omega t + \delta)}$$

Physical fields - take real parts

$$\vec{E}(\vec{r}, t) = \vec{E}_\omega e^{-k_2 z} \cos(k_1 z - \omega t)$$

$$\vec{B}(\vec{r}, t) = (\hat{z} \times \vec{E}_\omega) \frac{c|k|}{\omega} e^{-k_2 z} \cos(k_1 z - \omega t + \delta)$$

Conclusions

- 1) \vec{E} and $\vec{B} \perp \vec{k}$ transverse polarized
 - 2) $\vec{E} \perp \vec{B}$
 - 3) amplitude ratio $\frac{|\vec{B}|}{|\vec{E}|} = \frac{c|k|}{\omega} = \sqrt{\epsilon(\omega) \mu}$
 - 4) \vec{B} is shifted in phase with respect to \vec{E} by phase shift $\delta = \arctan(k_2/k_1)$
 - 5) waves decay as they propagate $e^{-k_2 z}$
- } consequence of complex $\epsilon(\omega)$

If $\epsilon_2 = 0$, i.e. $\epsilon(\omega)$ is real, and if $\epsilon > 0$, then $k_2 = 0 \Rightarrow$ no decay, no phase shift

consequences of frequency dependence of $\epsilon(\omega)$

- 6) $\vec{E}(t)$ and $\vec{D}(t)$ non locally related in time
- 7) waves of different ω travel with different $v_p = \omega/k_1$
- 8) dispersion - wave pulses do not travel with v_p and they spread as they propagate pulses travel with group velocity $v_g = \frac{d\omega}{dk}$ (see Quantum Mechanics discussion)

$v_g < v_p$ "normal dispersion"
 $v_g > v_p$ "anomalous dispersion"

$$\frac{1}{v_g} = \frac{dk_1}{d\omega} = \frac{d}{d\omega} \left[\frac{\omega}{c} n \right]$$

index of refraction

$$\frac{1}{v_g} = \frac{n}{c} + \frac{\omega}{c} \frac{dn}{d\omega} = \frac{1}{v_p} + \frac{\omega}{c} \frac{dn}{d\omega}$$

$$v_g = \frac{v_p}{1 + \frac{v_p}{c} \omega \frac{dn}{d\omega}}$$

⇒ when $\left\{ \begin{array}{l} \frac{dn}{d\omega} > 0, \quad v_g < v_p \text{ normal dispersion} \\ \frac{dn}{d\omega} < 0, \quad v_g > v_p \text{ anomalous dispersion} \end{array} \right.$

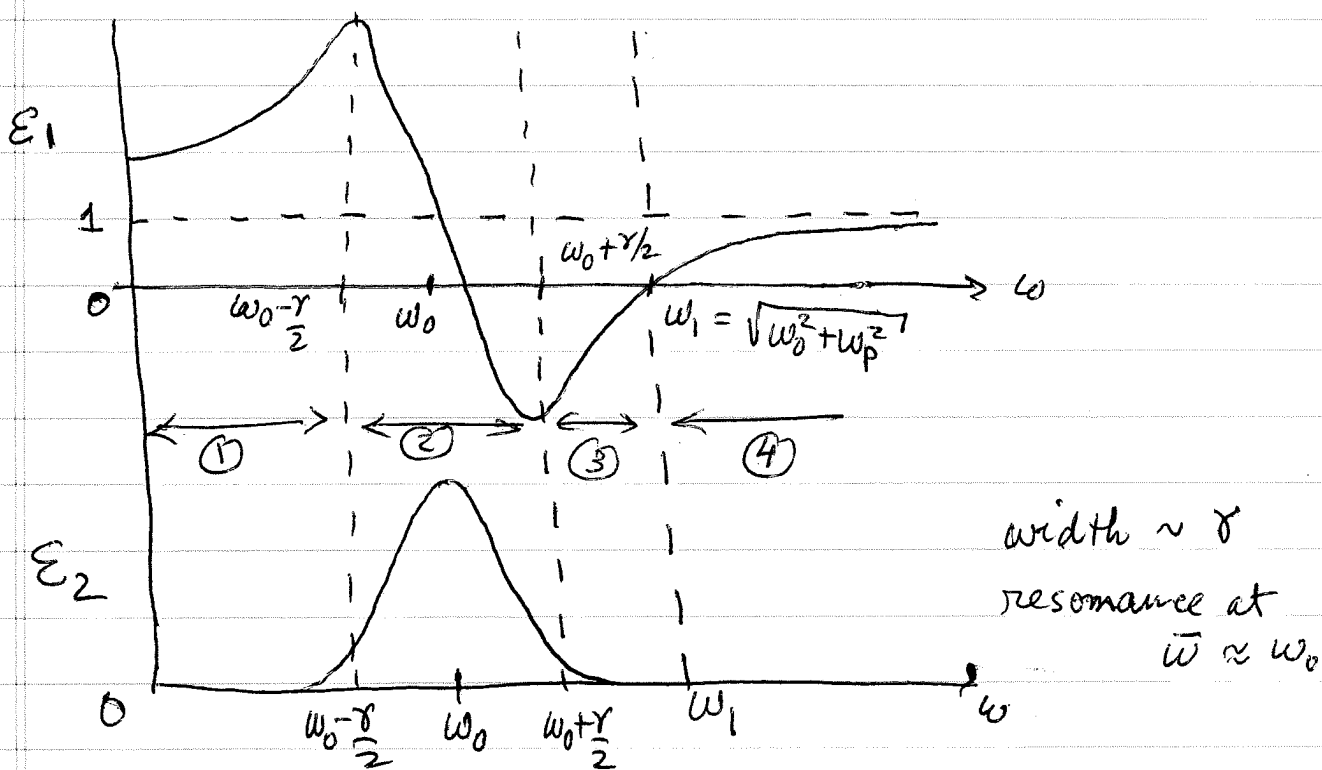
For our simple model: $\epsilon = 1 + 4\pi\chi \approx 1 + 4\pi\alpha$

$$\epsilon(\omega) = 1 + \frac{4\pi m e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - 2i\omega\gamma}$$

$$\epsilon_1 = 1 + \frac{4\pi m e^2}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}$$

$$\epsilon_2 = \frac{4\pi m e^2}{m} \frac{\omega \gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}$$

Define $\omega_p = \sqrt{\frac{4\pi m e^2}{m}}$ the "plasma frequency"



$$k = k_1 + ik_2 = \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\epsilon_1 + i\epsilon_2}$$

$$k^2 = k_1^2 - k_2^2 + 2ik_1 k_2 = \frac{\omega^2}{c^2} \mu (\epsilon_1 + i\epsilon_2)$$

equating real and imaginary pieces and solve for k_1 and k_2

$$k_1 = \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \sqrt{\epsilon_1^2 + \epsilon_2^2} + \frac{1}{2} \epsilon_1 \right]^{1/2}$$

$$k_2 = \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \sqrt{\epsilon_1^2 + \epsilon_2^2} - \frac{1}{2} \epsilon_1 \right]^{1/2}$$

Regions of different behavior

Regions ① and ④ - transparent propagation

$$\epsilon_1 > 0, \quad \epsilon_1 \gg \epsilon_2$$

$$k_1 \approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_1 \left(1 + \frac{1}{2} \left(\frac{\epsilon_2}{\epsilon_1} \right)^2 \right) + \frac{1}{2} \epsilon_1 \right]^{1/2}$$

$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\epsilon_1 + \frac{1}{4} \frac{\epsilon_2^2}{\epsilon_1} \right]^{1/2}$$

$$\approx \pm \frac{\omega}{c} \sqrt{\mu \epsilon_1} + \text{small correction}$$

$$k_2 \approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_1 \left(1 + \frac{1}{2} \left(\frac{\epsilon_2}{\epsilon_1} \right)^2 \right) - \frac{1}{2} \epsilon_1 \right]^{1/2}$$

$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{4} \frac{\epsilon_2^2}{\epsilon_1} \right]^{1/2} = k_1 \left(\frac{\epsilon_2}{2\epsilon_1} \right) \ll k_1$$

So $k_2 \ll k_1$ small attenuation
 \Rightarrow medium is transparent

Note: $v_p = \frac{\omega}{k_1} = \frac{c}{n} = \frac{c}{\sqrt{\epsilon_1 \mu}}$

in region ①, $\epsilon_1 > 1 \Rightarrow v_p < c$

in region ④, $\epsilon_1 < 1 \Rightarrow v_p > c!$

but $v_g < c$ always!

Region ② $\omega \approx \omega_0$ resonant absorption

$$\epsilon_2 \approx \frac{\omega_p^2}{\omega_0 \gamma} = \left(\frac{\omega_p}{\omega_0}\right)^2 \left(\frac{\omega_0}{\gamma}\right) \gg 1 \quad \text{for a sharp resonance with } \gamma \ll \omega_0$$
$$\epsilon_1 \approx 1$$

So $\epsilon_2 \gg \epsilon_1$

$$k_1 \approx \pm \frac{\omega \sqrt{\mu}}{c} \left[\frac{1}{2} \epsilon_2 \left(1 + \frac{1}{2} \left(\frac{\epsilon_1}{\epsilon_2} \right)^2 \right) + \frac{1}{2} \epsilon_1 \right]^{1/2}$$
$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_2 + \frac{1}{4} \frac{\epsilon_1^2}{\epsilon_2} + \frac{1}{2} \epsilon_1 \right]^{1/2}$$

$$k_1 \approx \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\frac{1}{2} \epsilon_2}$$

$$k_2 \approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_2 \left(1 + \frac{1}{2} \left(\frac{\epsilon_1}{\epsilon_2} \right)^2 \right) - \frac{1}{2} \epsilon_1 \right]^{1/2}$$
$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_2 + \frac{1}{4} \frac{\epsilon_1^2}{\epsilon_2} - \frac{1}{2} \epsilon_1 \right]^{1/2}$$
$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\frac{1}{2} \epsilon_2}$$

$k_1 \approx k_2$ strong attenuation

wave excites atoms at resonance \Rightarrow large atomic displacements \Rightarrow media absorbs most energy from the wave \Rightarrow wave decays rapidly, decreases factor $\frac{1}{e^{2\pi}}$ within one wavelength of propagation.

Region ③

$$\epsilon_1 < 0, \quad |\epsilon_1| \gg \epsilon_2$$

total reflection

width of region ③ is

$$\omega_1 - \omega_0 = \sqrt{\omega_0^2 + \omega_p^2} - \omega_0 \sim \omega_p \sim \sqrt{N}$$

increases with atomic density as $\omega_p \gg \omega_0$

$$k_1 \approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} |\epsilon_1| + \frac{1}{4} \frac{\epsilon_2^2}{|\epsilon_1|} + \frac{1}{2} \epsilon_1 \right]^{1/2}$$

↑ cancel as $|\epsilon_1| = -\epsilon_1$ ↓

$$k_1 \approx \pm \frac{\omega}{c} \sqrt{\mu |\epsilon_1|} \frac{\epsilon_2}{2|\epsilon_1|}$$

$$k_2 \approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} |\epsilon_1| + \frac{1}{4} \frac{\epsilon_2^2}{|\epsilon_1|} - \frac{1}{2} \epsilon_1 \right]^{1/2}$$
$$\approx \pm \frac{\omega}{c} \sqrt{\mu |\epsilon_1|}$$

$$\frac{k_2}{k_1} = \frac{2|\epsilon_1|}{\epsilon_2} \gg 1$$

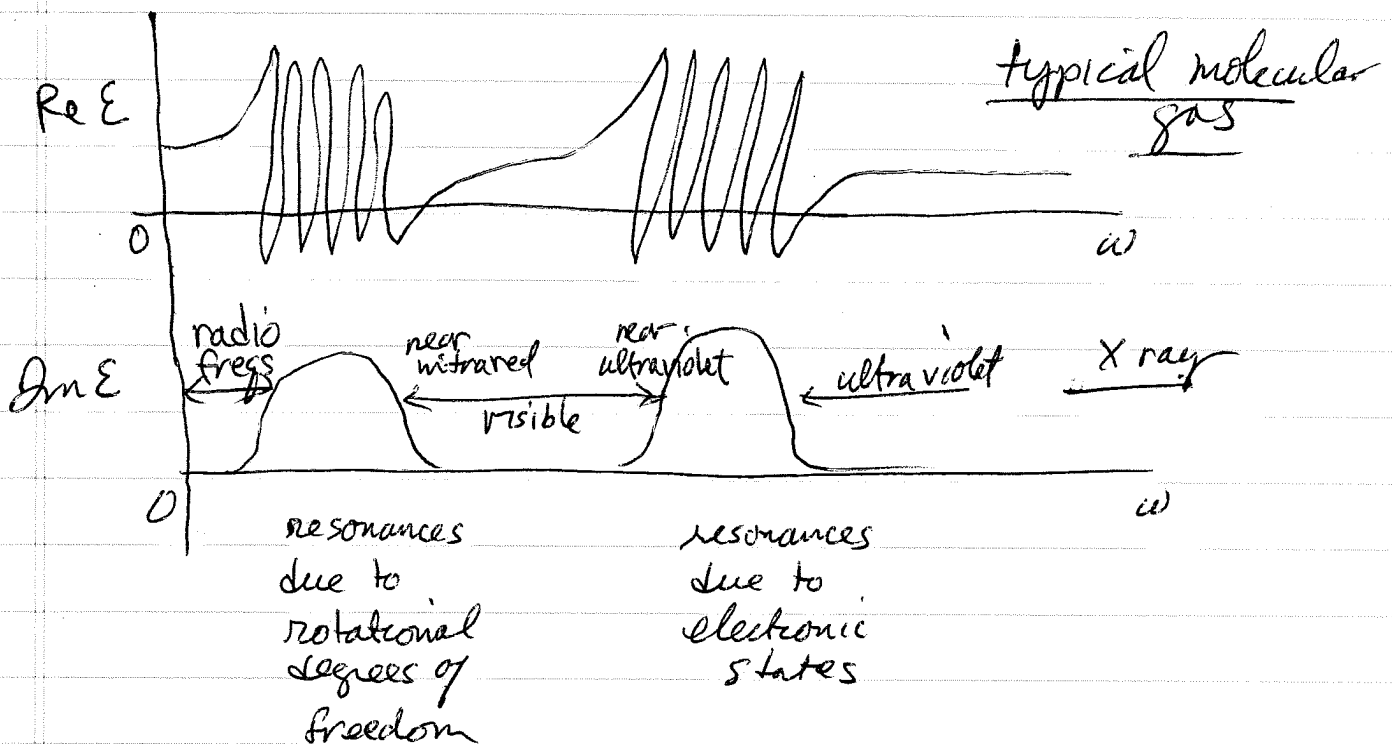
wave vector is almost pure imaginary
wave decays exponentially to zero in much less than one wavelength.

we will see this corresponds to total reflection
since $\omega \gg \omega_0$, we are not at resonance,
so material is not absorbing much energy from wave.
The strong attenuation is due to the destructive interference between the wave and the induced fields of the polarized atoms

Our single model had a single resonance at ω_0 .
 A more realistic model for molecules has many bands of resonances due to rotational, vibrational, and electronic modes of excitation.

$$\epsilon(\omega) = 1 + \omega_p^2 \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

where $\hbar\omega_i$ are spacings between energy levels with allowed electric dipole transitions



$$\omega_p = \sqrt{\frac{4\pi n e^2}{m}}$$

$$= 4.4 \times 10^{16} \sqrt{\frac{m}{M_A}} \text{ sec}^{-1}, \quad M_A = 6 \times 10^{23} / \text{cm}^3$$

For H_2O

$$\Rightarrow \hbar \omega_p = 185 \sqrt{\frac{m}{M_A}} \text{ eV}$$

$$\text{For } \text{H}_2\text{O} \quad \frac{m}{M_A} \sim 0.05$$

$$\hbar \omega_p \sim 40 \text{ eV}$$

$$\text{For typical metal } \frac{m}{M_A} \sim 0.1$$

$$\hbar \omega_p \sim 58 \text{ eV}$$

compared to $\hbar \omega_0 \sim \text{eV}$