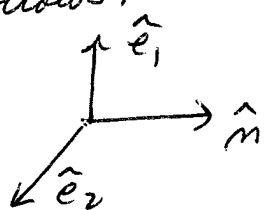


Polarization

Consider a transverse plane wave traveling in direction \hat{m} , i.e. $\vec{k} = k\hat{m}$. Define a right handed coordinate system as follows:



$$\begin{aligned} \hat{e}_1 \times \hat{e}_2 &= \hat{m} \\ \hat{m} \times \hat{e}_1 &= \hat{e}_2 \\ \hat{e}_2 \times \hat{m} &= \hat{e}_1 \end{aligned}$$

A general solution to Maxwell's equations for a transverse plane wave is then

$$\begin{aligned} \vec{E}(\vec{r}, t) &= \text{Re} \left\{ (E_1 \hat{e}_1 + E_2 \hat{e}_2) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right\} \\ \vec{H}(\vec{r}, t) &= \frac{c}{\omega \mu} \text{Re} \left\{ k \hat{m} \times (E_1 \hat{e}_1 + E_2 \hat{e}_2) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right\} \\ &= \frac{c}{\omega \mu} \text{Re} \left\{ k (E_1 \hat{e}_2 - E_2 \hat{e}_1) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right\} \end{aligned}$$

In general, k is complex
 $k = k_1 + ik_2 = |k| e^{i\delta}$, $\begin{cases} |k| = \sqrt{k_1^2 + k_2^2} \\ \delta = \arctan(k_2/k_1) \end{cases}$

So far we implicitly assumed that E_1 and E_2 are real constants. In this case

$$\begin{aligned} \vec{E}(\vec{r}, t) &= \vec{E}_\omega e^{-k_2 \hat{m} \cdot \vec{r}} \cos(k_1 \hat{m} \cdot \vec{r} - \omega t) \\ \vec{H}(\vec{r}, t) &= \vec{H}_\omega e^{-k_2 \hat{m} \cdot \vec{r}} \cos(k_1 \hat{m} \cdot \vec{r} - \omega t + \delta) \end{aligned}$$

where

$$\vec{E}_\omega \equiv E_1 \hat{e}_1 + E_2 \hat{e}_2 \quad \text{and} \quad \vec{H}_\omega \equiv \frac{c|k|}{\omega \mu} (E_1 \hat{e}_2 - E_2 \hat{e}_1)$$

are fixed vectors for all time and space.

In this case the directions of \vec{E} and \vec{H} remain fixed while the amplitudes oscillate in time and space. Such a plane wave is called a linearly polarized wave.

However there is nothing to prevent one from choosing a solution with E_1 and E_2 complex numbers,

$$E_1 = |E_1| e^{i\chi_1}, \quad E_2 = |E_2| e^{i\chi_2}$$

In this case one has

$$\begin{aligned} \vec{E}(\vec{r}, t) &= \text{Re} \left\{ |E_1| \hat{e}_1 e^{i(\vec{k} \cdot \vec{r} - \omega t + \chi_1)} + |E_2| \hat{e}_2 e^{i(\vec{k} \cdot \vec{r} - \omega t + \chi_2)} \right\} \\ &= e^{-k_2 \hat{m} \cdot \vec{r}} \left[|E_1| \hat{e}_1 \cos(k_1 \hat{m} \cdot \vec{r} - \omega t + \chi_1) \right. \\ &\quad \left. + |E_2| \hat{e}_2 \cos(k_1 \hat{m} \cdot \vec{r} - \omega t + \chi_2) \right] \end{aligned}$$

and

$$\begin{aligned} \vec{H}(\vec{r}, t) &= \frac{c|k|}{\omega\mu} \text{Re} \left\{ |E_1| \hat{e}_2 e^{i(\vec{k} \cdot \vec{r} - \omega t + \delta + \chi_1)} \right. \\ &\quad \left. - |E_2| \hat{e}_1 e^{i(\vec{k} \cdot \vec{r} - \omega t + \delta + \chi_2)} \right\} \\ &= \frac{c|k|}{\omega\mu} e^{-k_2 \hat{m} \cdot \vec{r}} \left[|E_1| \hat{e}_2 \cos(k_1 \hat{m} \cdot \vec{r} - \omega t + \delta + \chi_1) \right. \\ &\quad \left. - |E_2| \hat{e}_1 \cos(k_1 \hat{m} \cdot \vec{r} - \omega t + \delta + \chi_2) \right] \end{aligned}$$

Unless $\chi_1 = \chi_2$ we see that the components of \vec{E} and \vec{H} in directions \hat{e}_1 and \hat{e}_2 will oscillate out of phase with each other. Thus the directions of \vec{E} and \vec{H} will oscillate in time and space, as well as the amplitudes of \vec{E} and \vec{H} . The direction of \vec{E} and \vec{H} is no longer fixed.

We will see that this situation in general corresponds to an elliptically polarized wave!

General case E_1 and E_2 are complex constants

write $E_1 \hat{e}_1 + E_2 \hat{e}_2 \equiv \vec{U} e^{i\psi}$

where ψ is chosen so that $\vec{U} \cdot \vec{U}$ is real

- one can always do this since $\vec{U} \cdot \vec{U} = (E_1^2 + E_2^2) e^{-2i\psi}$
 so 2ψ is just the phase of the complex $E_1^2 + E_2^2$

\vec{U} is a complex vector $\Rightarrow \vec{U} = \vec{U}_a + i\vec{U}_b$

with \vec{U}_a and \vec{U}_b real vectors

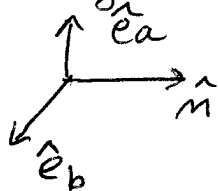
Since $\vec{U} \cdot \vec{U}$ is real $\Rightarrow \vec{U}_a \cdot \vec{U}_b = 0$

so $\vec{U}_a \perp \vec{U}_b$ orthogonal

Let \hat{e}_a be the unit vector in direction of \vec{U}_a

so $\vec{U}_a = U_a \hat{e}_a$ with $U_a = |\vec{U}_a|$

Let $\hat{e}_b = \hat{m} \times \hat{e}_a$ so that $\{\hat{m}, \hat{e}_a, \hat{e}_b\}$ are a right handed coordinate system



Then $\vec{U}_b = \pm U_b \hat{e}_b$ where $U_b = |\vec{U}_b|$

since $\vec{U}_b \perp \vec{U}_a$ and both are \perp to \hat{m} .

It is (+) if \vec{U}_b is parallel to \hat{e}_b and
 it is (-) if \vec{U}_b is antiparallel to \hat{e}_b .

In this representation we have

$$\begin{aligned}\vec{E}(\vec{r}, t) &= \text{Re} \left\{ \vec{U} e^{i\psi} e^{-i(\vec{k} \cdot \vec{r} - \omega t)} \right\} \\ &= e^{-k_z \hat{m} \cdot \vec{r}} \text{Re} \left\{ U_a \hat{e}_a e^{i(k_1 \hat{m} \cdot \vec{r} - \omega t + \psi)} \right. \\ &\quad \left. \pm U_b \hat{e}_b (\pm i) e^{i(k_1 \hat{m} \cdot \vec{r} - \omega t + \psi)} \right\} \\ &= e^{-k_z \hat{m} \cdot \vec{r}} \left\{ U_a \hat{e}_a \cos(\Phi + \psi) \mp U_b \hat{e}_b \sin(\Phi + \psi) \right\}\end{aligned}$$

where we write $\Phi \equiv k_1 \hat{m} \cdot \vec{r} - \omega t$

Let's define

$$\begin{aligned}e^{-k_z \hat{m} \cdot \vec{r}} U_a &\rightarrow U_a \\ e^{-k_z \hat{m} \cdot \vec{r}} U_b &\rightarrow U_b\end{aligned}$$

so we don't have to keep writing the constant attenuation factor that is a common factor of all components of \vec{E} .

Then define E_a and E_b as the components of \vec{E} in the directions \hat{e}_a and \hat{e}_b respectively.

$$E_a = U_a \cos(\Phi + \psi)$$

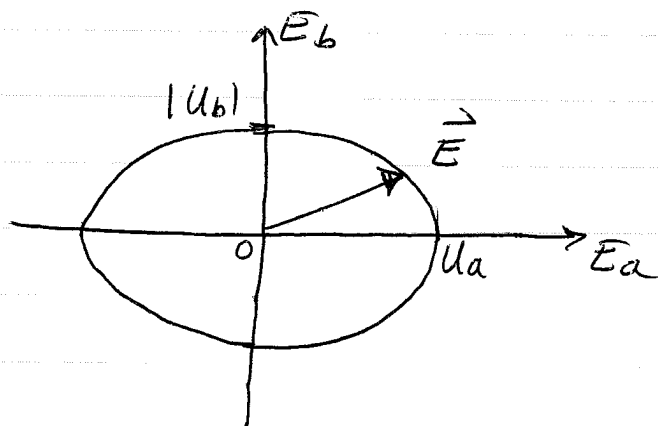
$$E_b = \mp U_b \sin(\Phi + \psi)$$

This then gives

$$\left(\frac{E_a}{U_a}\right)^2 + \left(\frac{E_b}{U_b}\right)^2 = \cos^2(\Phi + \psi) + \sin^2(\Phi + \psi) = 1$$

This is just the equation for an ellipse

with semi-axes of lengths U_a and U_b , oriented in the directions of \hat{e}_a and \hat{e}_b .



\Rightarrow At a fixed position \vec{r} , the tip of the vector \vec{E} will trace out the above ellipse as the time increases by one period of oscillation $2\pi/\omega$.

For (+), i.e. $\vec{U}_b = U_b \hat{e}_b$, \vec{E} goes around the ellipse counterclockwise as t increases

For (-), i.e. $\vec{U}_b = -U_b \hat{e}_b$, \vec{E} goes around the ellipse clockwise as t increases

Such a wave is said to be elliptically polarized

Special cases

① $U_a = 0$ or $U_b = 0$
the wave is linearly polarized

$$(2) \quad U_a = U_b$$

The tip of \vec{E} traces out a ~~circle~~ circle as t increases, the wave is circularly polarized.

The (+) case is said to have right handed circular polarization

The (-) case is said to have left handed circular polarization

One can define circular polarization basis vectors

$$\hat{e}_+ \equiv \frac{\hat{e}_a + i\hat{e}_b}{\sqrt{2}} \quad \hat{e}_- \equiv \frac{\hat{e}_a - i\hat{e}_b}{\sqrt{2}}$$

with \hat{e}_a and \hat{e}_b orthogonal.

A wave with ^{complex} amplitude $\vec{E}_w = E \hat{e}_+$ is right handed circularly polarized.

A wave with complex amplitude $\vec{E}_w = E \hat{e}_-$ is left handed circularly polarized.

Just as the general case can always be written as a superposition of two orthogonal linearly polarized waves, i.e.

$$\vec{E}_w = E_1 \hat{e}_1 + E_2 \hat{e}_2$$

one can also always write the general case as a superposition of a left handed and a right handed circularly polarized wave

$$\vec{U} = \vec{U}_a + i\vec{U}_b = U_a \hat{e}_a \pm iU_b \hat{e}_b$$

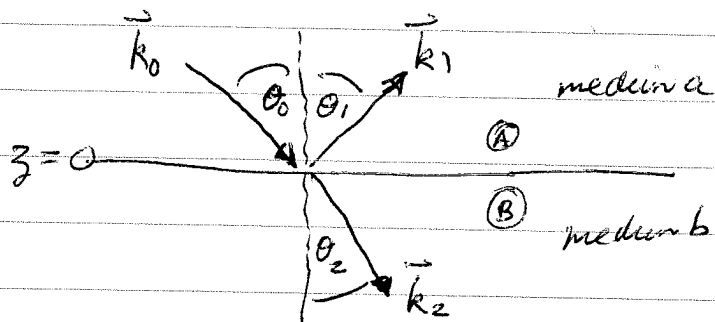
$$= \left(\frac{U_a + U_b}{\sqrt{2}} \right) \hat{e}_{\pm} + \left(\frac{U_a - U_b}{\sqrt{2}} \right) \hat{e}_{\mp}$$

(rearrange substituted in for \hat{e}_{\pm} and expand, to see that this is so)

\Rightarrow An elliptically polarized wave can be written as a superposition of circularly polarized waves

As a special case of the above (if $U_a = 0$ or $U_b = 0$) a linearly polarized wave can always be written as a superposition of circularly polarized waves.

Reflection & Transmission of waves at Interfaces



consider wave propagating from medium A into medium B.

for simplicity assume ϵ_a is real and positive, ϵ_b may be complex
 μ_a and μ_b are real and constant

\vec{k}_0 is incident wave, $\theta_0 =$ angle of incidence

\vec{k}_1 is reflected wave, $\theta_1 =$ angle of reflection

\vec{k}_2 is the transmitted or "refracted" wave, $\theta_2 =$ angle of refraction

let each wave be given by

$$\vec{F}_n(\vec{r}, t) = \vec{F}_n e^{i(\vec{k}_n \cdot \vec{r} - \omega_n t)}$$

where \vec{F}_n can be either \vec{E}_n or \vec{H}_n for the electric or magnetic component of the wave

boundary condition: tangential component \vec{E} must be continuous at $z=0$. If \hat{x} is a vector in xy plane, and we consider $\vec{r}=0$, then

$$\Rightarrow \hat{x} \cdot \vec{E}_0 e^{-i\omega_0 t} + \hat{x} \cdot \vec{E}_1 e^{-i\omega_1 t} = \hat{x} \cdot \vec{E}_2 e^{-i\omega_2 t}$$

must be true for all time. can only happen if

$$\boxed{\omega_0 = \omega_1 = \omega_2 \equiv \omega} \quad \text{all frequencies are equal}$$

Now consider the same boundary condition for \vec{p} a position vector in the xy plane at $z=0$. Since ω 's are all equal we can cancel out the common $e^{-i\omega t}$ factors to get

$$\hat{x} \cdot \vec{E}_0 e^{i\vec{k}_0 \cdot \vec{p}} + \hat{x} \cdot \vec{E}_1 e^{i\vec{k}_1 \cdot \vec{p}} = \hat{x} \cdot \vec{E}_2 e^{i\vec{k}_2 \cdot \vec{p}}$$

this must be true for all \vec{p} . Can only happen if the projections of the \vec{k}_n in the xy plane are all equal

$$\boxed{\begin{aligned} k_{0x} &= k_{1x} = k_{2x} \\ k_{0y} &= k_{1y} = k_{2y} \end{aligned}}$$

only z components \vec{k} vectors can be different

Choose coord system as in diagram so that all \vec{k} vectors lie in the xz plane (y is out of page)

Since ϵ_a is real and positive, \vec{k}_0 and \vec{k}_1 are real vectors

$$k_{0x} = k_{1x} \Rightarrow |\vec{k}_0| \sin \theta_0 = |\vec{k}_1| \sin \theta_1$$

$$\text{since } k_0^2 = \frac{\omega^2}{c^2} \mu \epsilon_a \quad \text{and } k_1^2 = \frac{\omega^2}{c^2} \mu \epsilon_a$$

$$\text{then } |\vec{k}_0| = |\vec{k}_1| \quad \text{so} \quad \sin \theta_0 = \sin \theta_1$$

$$\boxed{\theta_0 = \theta_1}$$

angle of incidence = angle of reflection