i) Consider region of "total reflection"

$\Rightarrow$ $\left.\begin{array}{l} \mathrm{Im}\,\epsilon_b = \epsilon_{b2} \approx 0 \\ \mathrm{Re}\,\epsilon_b = \epsilon_{b1} < 0 \end{array}\right\} \Rightarrow \vec{k}_2 = i\vec{K}_2$ where $\vec{K}_2$ is real

ie $\vec{k}_2$ pure imaginary

$$\Rightarrow \quad R_\perp = \left| \frac{\mu_b k_{0z} - i\mu_a K_{2z}}{\mu_b k_{0z} + i\mu_a K_{2z}} \right|^2$$

$$R_\parallel = \left| \frac{\epsilon_b k_{0z} - i\epsilon_a K_{2z}}{\epsilon_b k_{0z} + i\epsilon_a K_{2z}} \right|^2$$

both are of the form $\left|\frac{a-ib}{a+ib}\right|^2 = 1$ when $a, b$ real

$$\Rightarrow R_\perp = R_\parallel = 1$$

confirms that the material is completely reflecting

ii) Next consider when medium B is transparent

$\epsilon_b$ is real and $\epsilon_b > 0$

$$k_{0z} = \frac{\omega}{c}\sqrt{\mu_a \epsilon_a}\cos\theta_0 = \frac{\omega}{c} n_a \cos\theta_0$$
$$k_{2z} = \frac{\omega}{c}\sqrt{\mu_b \epsilon_b}\cos\theta_2 = \frac{\omega}{c} n_b \cos\theta_2$$

Snell's law holds so $n_a \sin\theta_0 = n_b \sin\theta_2$

can write $R_\perp$ and $R_\parallel$ as functions of $\theta_0$

for simplicity take $\mu_a = \mu_b = 1$

(1) $R_\perp = \left(\frac{n_a \cos\theta_0 - n_b \cos\theta_2}{n_a \cos\theta_0 + n_b \cos\theta_2}\right)^2 = \left(\frac{\cos\theta_0 - \left(\frac{\sin\theta_0}{\sin\theta_2}\right)\cos\theta_2}{\cos\theta_0 + \left(\frac{\sin\theta_0}{\sin\theta_2}\right)\cos\theta_2}\right)^2$

$$= \left(\frac{\sin\theta_2 \cos\theta_0 - \sin\theta_0 \cos\theta_2}{\sin\theta_2 \cos\theta_0 + \sin\theta_0 \cos\theta_2}\right)^2$$

$$R_\perp = \left(\frac{\sin(\theta_0 - \theta_2)}{\sin(\theta_0 + \theta_2)}\right)^2$$

for $\theta_0 = 0$, ie normal incidence, $\theta_2 = 0$

$\Rightarrow R_\perp = \left(\frac{n_a - n_b}{n_a + n_b}\right)^2$ if $n_a = n_b$, no reflection!
(not surprising!)

(2) $R_\parallel = \left(\frac{\epsilon_b n_a \cos\theta_0 - \epsilon_a n_b \cos\theta_2}{\epsilon_b n_a \cos\theta_0 + \epsilon_a n_b \cos\theta_2}\right)^2$ use $\sqrt{\epsilon_b} = n_b$
$\sqrt{\epsilon_a} = n_a$

$$= \left(\frac{n_b \cos\theta_0 - n_a \cos\theta_2}{n_b \cos\theta_0 + n_a \cos\theta_2}\right)^2$$

$$= \left(\frac{\cos\theta_0 - \left(\frac{\sin\theta_2}{\sin\theta_0}\right)\cos\theta_2}{\cos\theta_0 + \left(\frac{\sin\theta_2}{\sin\theta_0}\right)\cos\theta_2}\right)^2$$

$$= \left(\frac{\sin\theta_0 \cos\theta_0 - \sin\theta_2 \cos\theta_2}{\sin\theta_0 \cos\theta_0 + \sin\theta_2 \cos\theta_2}\right)^2$$

$R_\parallel = \left(\frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)}\right)^2$ ← after some algebra!

for $\theta_0 = 0$, then $\theta_2 = 0$

$$R_{\parallel} = \left(\frac{\epsilon_b n_a - \epsilon_a n_b}{\epsilon_b n_a + \epsilon_a n_b}\right)^2 = \left(\frac{n_b - n_a}{n_b + n_a}\right)^2 \quad \text{same as } R_{\perp}$$

So for $\theta_0 = 0$, $R_{\parallel} = R_{\perp}$ – this must be so since for $\theta_0 = 0$ there is no distinction between the $\perp$ and $\parallel$ cases

If $n_b = n_a$, $R_{\perp} = R_{\parallel} = 0$ no reflective wave

When $\theta_0 + \theta_2 = \pi/2$, then $\tan(\theta_0 + \theta_2) \to \infty$ and $R_{\parallel} = 0$

This occurs at an angle of incidence known as Brewster's angle $\theta_B$, determined by

$$\underset{\substack{\uparrow \\ \theta_0}}{n_a \sin \theta_B} = n_b \underset{\substack{\uparrow \\ \theta_2}}{\sin\left(\frac{\pi}{2} - \theta_B\right)} = n_b \cos \theta_B$$

$$\Rightarrow \boxed{\tan \theta_B = \frac{n_b}{n_a}}$$

For incident wave at $\theta_B$, reflected wave always has $\vec{E}_1 \perp$ plane of incidence, since $R_{\parallel} = 0$. If incoming wave has $\vec{E}_0 \parallel$ plane of incidence, then it gets completely transmitted. If $\vec{E}_0$ in general direction, reflected wave is always linearly polarized with $\vec{E}_1 \perp$ plane of incidence. – This is one method to create polarized light wave.

# Kramers - Kronig Relation

We saw that $\vec{P}_\omega = \alpha(\omega)\vec{E}_\omega$

<u>Causal</u> response ie $\tilde{\alpha}(t) = 0$ for $t < 0$

$\Rightarrow$ $\alpha(\omega)$ has no poles in upper half of complex $\omega$ plane (UHP)

For any complex $\bar{\omega}$ in upper half of complex $\omega$ plane,

$$\alpha(\bar{\omega}) = \frac{1}{2\pi i}\oint \frac{\alpha(\omega')}{\omega'-\bar{\omega}}\,d\omega'$$ since no poles of $\alpha$ in UHP

contour along real axis, closed at infinity in UHP. The closing ~~loop~~ semicircle at infinity gives no contribution assuming $\alpha(\omega)$ decays quickly enough as $|\omega| \to \infty$

$$\alpha(\bar{\omega}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega'-\bar{\omega}}$$

Now consider $\bar{\omega} = \omega + i\delta$ where $\omega$ and $\delta$ are real and $\delta \to 0$

$$\alpha(\omega) = \lim_{\delta\to 0} \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega'-\omega-i\delta}$$

Now $$\frac{1}{\omega'-\omega-i\delta} = P\left(\frac{1}{\omega'-\omega}\right) + i\pi\delta(\omega'-\omega)$$

principle part

$$\Rightarrow \alpha(\omega) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{\alpha(\omega')\, d\omega'}{\omega' - \omega}$$

$$\Rightarrow \boxed{\begin{aligned} \mathrm{Re}\, \alpha(\omega) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Im}\, \alpha(\omega')\, d\omega'}{\omega' - \omega} \\ \mathrm{Im}\, \alpha(\omega) &= \frac{-1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Re}\, \alpha(\omega')\, d\omega'}{\omega' - \omega} \end{aligned}}$$

Kramer Kronig relations

If know Re $\alpha$ <u>or</u> Im $\alpha$ can reconstruct full complex $\alpha$

True for any causal response function

# Radiation From Moving Charges

In Lorentz gauge $\quad \frac{1}{c}\frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$

$$\left.\begin{aligned} \nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} &= -4\pi\rho \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} &= -\frac{4\pi}{c}\vec{j} \end{aligned}\right\} \text{ wave equation with source}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$
$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

If we can solve wave equation with source (inhomogeneous wave equation) then we are in principle done! To do this we want to find the Green's function for the wave equation

Recall from statics: $\quad \nabla^2\phi = -4\pi\rho$

Greens function satisfies $\quad \nabla^2 G(\vec{r}) = -4\pi\,\delta(\vec{r})$

then $\quad \phi(\vec{r}) = \int d^3r'\, G(\vec{r}-\vec{r}')\,\rho(\vec{r}') \; + \phi_0$

solution for infinite volume that vanishes as $r\to\infty$ is

$$G(\vec{r}-\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} \qquad\qquad \nabla^2\phi_0 = 0$$

For wave equation we want solution to

$$\nabla^2 G(\vec{r},t;\vec{r}',t') - \frac{1}{c^2}\frac{\partial^2 G(\vec{r}t;\vec{r}'t')}{\partial t^2} = -4\pi\,\delta(\vec{r}-\vec{r}')\,\delta(t-t')$$

Then we will have
$$\begin{cases} \phi(\vec{r},t) = \int dt' \int d^3r'\, G(\vec{r},t;\vec{r}'t')\,\rho(\vec{r}',t') + \phi_0 \\ \vec{A}(\vec{r},t) = \frac{1}{c}\int dt' \int d^3r'\, G(\vec{r}t;\vec{r}'t')\,\vec{j}(\vec{r}',t') + \vec{A}_0 \end{cases}$$

where $\nabla^2\Phi_0 - \frac{1}{c^2}\frac{\partial^2\Phi_0}{\partial t^2} = 0$ similarly for $\vec{A}_0$

$\Phi_0$ and $\vec{A}_0$ could describe an incoming wave for example

To construct the Green's function.

For <u>infinite space</u> (but not, for example, inside a cavity)

$G(\vec{r}t; \vec{r}'t') = G(\vec{r}-\vec{r}', t-t')$

express as Fourier transform

$$G(\vec{r}, t) = \int \frac{d^3k\, d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r}, t) = \int \frac{d^3k\, d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\parallel \qquad = \int \frac{d^3k\, d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) \left[-k^2 + \frac{\omega^2}{c^2}\right] e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$-4\pi\delta(\vec{r})\delta(t) = -4\pi \int \frac{d^3k\, d\omega}{(2\pi)^4} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

equate Fourier amplitudes

$$\Rightarrow \quad \left[-k^2 + \frac{\omega^2}{c^2}\right] \tilde{G}(\vec{k}, \omega) = -4\pi$$

$$\boxed{\tilde{G}(\vec{k}, \omega) = \frac{4\pi c^2}{k^2c^2 - \omega^2}} \qquad \text{when } \omega^2 \neq c^2k^2$$

$$G(\vec{r}, t) = \int \frac{d^3k\, d\omega}{(2\pi)^4} \frac{4\pi c^2}{k^2c^2 - \omega^2} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

↑ poles at $\omega = \pm ck$

In evaluating the $\omega$ integral we have to know how to treat the poles on the real axis so that $G(\vec{r},t)$ will have the desired behavior.

What we want is for $G(\vec{r},t)$ to be <u>causal</u>, ie $G(\vec{r},t)=0$ for $t<0$, so $\phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ depend only on the values of the sources at <u>earlier</u> times $t'<t$.

$$\int d^3k\, e^{i\vec{k}\cdot\vec{r}}\,\tilde{G}(\vec{k},\omega) = 2\pi\int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk\, k^2\, e^{ikr\cos\theta}\,\tilde{G}(k,\omega)$$

$\mu \equiv \cos\theta$

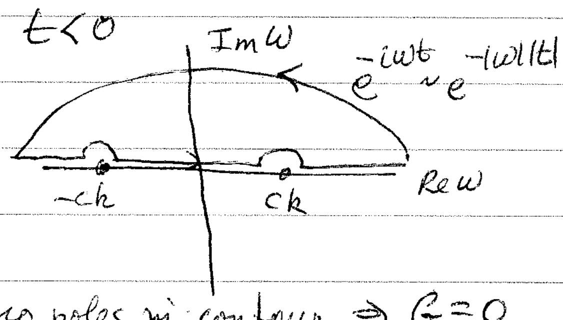
$$= 2\pi \int_{-1}^{1} d\mu \int_0^{\infty} dk\, k^2\, e^{ikr\mu}\,\tilde{G}(k,\omega)$$

$$= 4\pi \int_0^{\infty} dk\, k^2\, \frac{\sin kr}{kr}\,\tilde{G}(k,\omega)$$

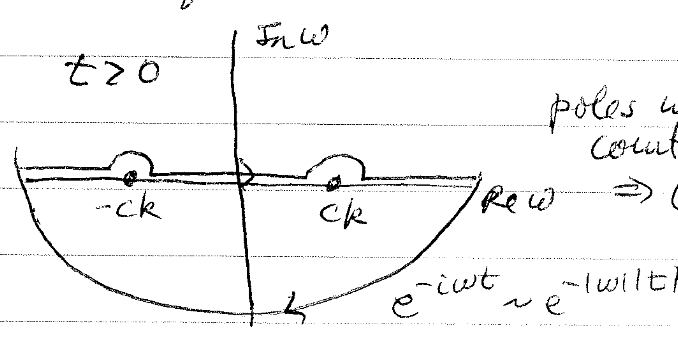
$$G(\vec{r},t) = -\frac{c^2}{\pi^2}\int_0^{\infty} dk\, k^2\, \frac{\sin kr}{kr}\int_C \frac{e^{-i\omega t}}{(\omega+ck)(\omega-ck)}\,d\omega$$

contour along real axis, but deformed to go around the poles

for $t<0$, $e^{-i\omega t}$ will decay exponentially fast for large $|\omega|$ in the <u>upper half complex</u> (UHP) $\omega$ plane $\Rightarrow$ can close contour in UHP for $t<0$. If we want $G=0$ for $t<0$, there should therefore be no poles in UHP. The contour $C$ we want is therefore:




no poles in contour $\Rightarrow G=0$




poles within contour $\Rightarrow G \neq 0$

with this convention for the contour $C$ we can evaluate the $\omega$-integral using Cauchy's residue theorem

$$\int \frac{e^{-i\omega t}\, d\omega}{(\omega+ck)(\omega-ck)} = -2\pi i \left[ \frac{e^{-ickt}}{2ck} - \frac{e^{ickt}}{2ck} \right]$$

$$= -\frac{2\pi \sin(ckt)}{ck}$$

$$G(\vec{r},t) = \frac{2c}{\pi r}\int_0^\infty dk \sin(kr)\sin(ckt) = \frac{c}{\pi r}\int_{-\infty}^{\infty} dk \frac{(e^{ikr}-e^{-ikr})(e^{ickt}-e^{-ickt})}{(-4)}$$

$$= -\frac{c}{2r}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left\{ e^{i(r+ct)k} + e^{-i(r+ct)k} - e^{i(r-ct)k} - e^{-i(r-ct)k} \right]$$

each integral would give a $\delta$-function, but for 1st two terms $\delta(r+ct)=0$ since here $t>0$ (by definition) and $r=|\vec{r}| \geq 0$ so the arguement will never vanish.

$$\boxed{G(\vec{r},t) = \frac{c}{r}\,\delta(r-ct) = \frac{\delta(t-r/c)}{r}}$$

using $\delta(ax) = \frac{1}{a}\delta(x)$

$$G(\vec{r}t,\vec{r}'t') = \begin{cases} \dfrac{\delta\left(t-t' - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} & t-t'>0 \\ 0 & t-t'<0 \end{cases}$$

Green's function for wave equation in free space

$G \neq 0$ only on "light cone" that ~~connects~~ emanates from $(r',t')$, ie when $|\vec{r}-\vec{r}'| = c(t-t')$.

Signal from source at $(r',t')$ travels with $c$