$$\phi(\vec{r},t) = \phi_0(\vec{r},t) + \int d^3r' \int_{-\infty}^{t} dt' \, \frac{\delta\left(t - t' - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} \, \rho(\vec{r}',t')$$

$$A(\vec{r},t) = \vec{A}_0(\vec{r},t) + \int \frac{d^3r'}{c} \int_{-\infty}^{t} dt' \, \frac{\delta\left(t - t' - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} \, \vec{j}(\vec{r}',t')$$

Apply to a <u>single moving point charge</u>

$$\rho(\vec{r},t) = q\,\delta(\vec{r} - \vec{r}_0(t))$$

$$\vec{j}(\vec{r},t) = q\,\vec{v}(t)\,\delta(\vec{r}-\vec{r}_0(t)) \qquad \text{where } \vec{v}(t) = \frac{d\vec{r}_0}{dt}$$

Then

$$\phi(\vec{r},t) = q \int dt' \, \frac{\delta\left(t - t' - \frac{1}{c}|\vec{r}-\vec{r}_0(t')|\right)}{|\vec{r}-\vec{r}_0(t')|}$$

because of the $\vec{r}_0(t')$ in the argument of the $\delta()$ function the t' dependence is not of the simple form $t'-t_0$.

We can write

$$g(t') \equiv t' + \frac{1}{c}\,|\vec{r}-\vec{r}_0(t')|$$

then

$$\phi(\vec{r},t) = q \int dt' \, \frac{\delta(t - g(t'))}{|\vec{r}-\vec{r}_0(t')|}$$

$$= q \int \frac{\delta(t-g(t'))}{|\vec{r}-\vec{r}_0(t')|} \, dg \left(\frac{dt'}{dg}\right)$$

$$= \frac{q}{|\vec{r}-\vec{r}_0(t')|} \, \frac{1}{(dg/dt')} \Bigg|_{t' \text{ such that } g(t') = t}$$

$$g(t') = t' + \frac{1}{c}\sqrt{[x - x_0(t')]^2 + [y - y_0(t')]^2 + [z - z_0(t')]^2}$$

$$\frac{dg}{dt'} = 1 + \frac{1}{c\,|r - r_0(t')|}\left\{[x - x_0(t')]\left(-\frac{dx_0}{dt'}\right) + \cdots\right\}$$

$$= 1 - \frac{1}{c}\,\hat{n}(t') \cdot \vec{v}(t')$$

where $\hat{n}(t') = \frac{\vec{r} - \vec{r}_0(t')}{|\vec{r} - \vec{r}_0(t')|}$ unit vector pointing from $\vec{r}_0(t')$ to $\vec{r}$

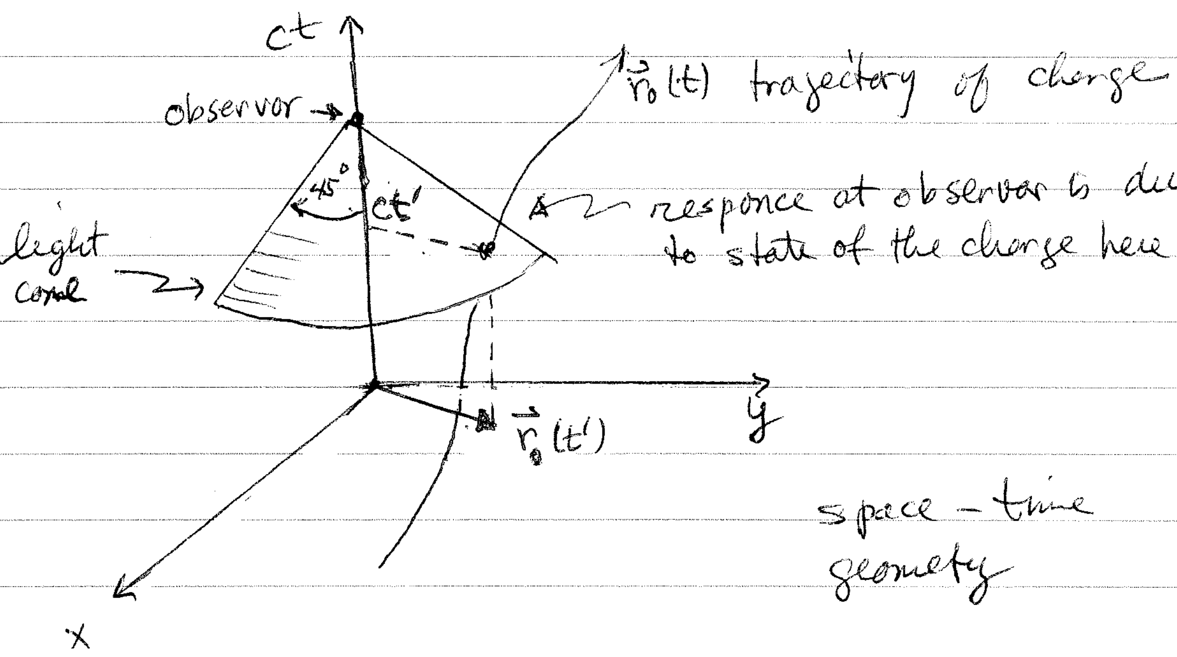
$$\phi(\vec{r}, t) = \frac{q}{|\vec{r} - \vec{r}_0(t')|\,[1 - \frac{1}{c}\hat{n}(t') \cdot \vec{v}(t')]}$$

$$\vec{A}(\vec{r}, t) = \frac{q\,\vec{v}(t')/c}{|\vec{r} - \vec{r}_0(t')|\,[1 - \frac{1}{c}\hat{n}(t') \cdot \vec{v}(t')]}$$

Liénard-Wiechert Potentials

where t' is determined by the condition

$$t - t' = \frac{1}{c}|\vec{r} - \vec{r}_0(t')|$$

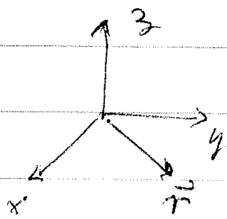


space-time geometry

For charge moving with constant velocity along $\hat{z}$

$$\vec{r}_0(t) = vt\,\hat{z} \qquad \vec{v} = \frac{d\vec{r}_0}{dt} = v\,\hat{z}$$

For observer at position $\vec{r}$ (in xy plane), time t, the fields will be determined by the charge at time t' such that



$$t - t' - \frac{|\vec{r} - \vec{r}_0(t')|}{c} = 0$$

$$t - t' - \frac{\sqrt{r^2 + v^2 t'^2}}{c} = 0$$

$$(t - t')^2 = t^2 + t'^2 - 2tt' = \frac{r^2 + v^2 t'^2}{c^2}$$

$$(1 - v^2/c^2)\,t'^2 - 2tt' + t^2 - \frac{r^2}{c^2} = 0$$

let $\gamma = (1 - v^2/c^2)^{-1/2}$

$$t'^2 - 2\gamma^2 t\,t' + \gamma^2 \frac{(c^2t^2 - r^2)}{c^2} = 0$$

$$t' = \gamma^2 t \pm \sqrt{\gamma^4 t^2 - \gamma^2 t^2 + \gamma^2 \frac{r^2}{c^2}}$$

$$= \gamma^2 t \pm \sqrt{\gamma^2\left(\gamma^2 t^2 - t^2 + \frac{r^2}{c^2}\right)}$$

$$\gamma^2 - 1 = \frac{1}{1 - v^2/c^2} - 1 = \frac{v^2/c^2}{1 - v^2/c^2} = \gamma^2 \frac{v^2}{c^2}$$

$$= \gamma^2 t \pm \gamma \sqrt{t^2 \gamma^2 \left(\frac{v^2}{c^2}\right) + \frac{r^2}{c^2}}$$

$$t' = \gamma^2 t \pm \frac{\gamma^2}{c} \sqrt{v^2 t^2 + \frac{r^2}{\gamma^2}}$$

consider $t=0$. Solution should give $t'<0$

$\Rightarrow$ $(-)$ sign is the solution we want

$$t' = \gamma^2 t - \frac{\gamma^2}{c}\sqrt{v^2 t^2 + \frac{r^2}{\gamma^2}}$$

$$\phi(\vec{r},t) = \frac{q}{|\vec{r}-\vec{r}_0(t')|\left[1-\frac{1}{c}\hat{n}(t')\cdot\vec{v}\right]}$$

$$|\vec{r}-\vec{r}_0(t')| = \sqrt{r^2+v^2t'^2} = c(t-t')$$ from condition that determines t'

$$(\vec{r}-\vec{r}_0(t'))\cdot\vec{v} = -\vec{r}_0(t')\cdot\vec{v}$$ for $\vec{v} = v\hat{z}$

$$= -v^2 t'$$ $\vec{r}$ in xy plane

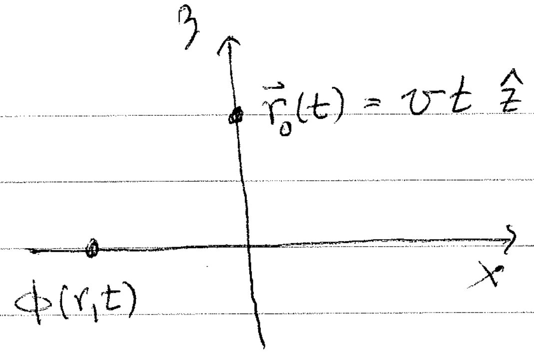
$$\phi(\vec{r},t) = \frac{q}{c(t-t')\left[1+\frac{v^2t'}{c^2(t-t')}\right]}$$

$$= \frac{q}{c(t-t') + \frac{v^2t'}{c}} = \frac{q}{c\left[t-\left(1-\frac{v^2}{c^2}\right)t'\right]}$$

$$= \frac{q}{c\left(t-\frac{t'}{\gamma^2}\right)} = \frac{q}{c\,\frac{1}{c}\sqrt{v^2t^2+r^2/\gamma^2}}$$

$$\left.\begin{aligned}\phi(\vec{r},t) &= \frac{q}{\sqrt{v^2t^2+r^2/\gamma^2}}\\[1em] \vec{A}(\vec{r},t) &= \frac{q\vec{v}}{c\sqrt{v^2t^2+r^2/\gamma^2}}\end{aligned}\right\}$$

solutions for $\vec{r}$ in xy plane when charge passes through xy plane at $t=0$



at x
~~potential from q at $vt\,\hat{z}$~~

potential at pt $\vec{r}$ in xy plane at time t, when charge is at $\vec{r}_0 = vt\,\hat{z}$, looks almost like static Coulomb potential, which would be $\dfrac{q}{\sqrt{r^2+v^2t^2}}$

But instead, it is

$$\frac{q}{\sqrt{v^2t^2+\left(\frac{r}{\gamma}\right)^2}}$$

looks like the transverse direction has contracted by a factor γ!

Such considerations led Lorentz to discover the Lorentz transformation, <u>before</u> Einstein proposed his theory of special relativity

Radiation from a Localized Oscillating Source

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int d^3r'\,dt'\ \frac{\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}\ \vec{j}(\vec{r}',t')$$

For pure harmonic oscillation in current

$$\vec{j}(\vec{r},t) = \mathrm{Re}\left\{\vec{j}_\omega(\vec{r})\,e^{-i\omega t}\right\}$$

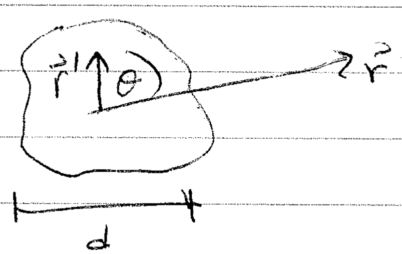
$$\Rightarrow\ \vec{A}(r,t) = \mathrm{Re}\left\{\vec{A}_\omega(\vec{r})\,e^{-i\omega t}\right\}$$

$$\Rightarrow\ \vec{A}_\omega(\vec{r})e^{-i\omega t} = \frac{1}{c}\int d^3r'\ \vec{j}_\omega(\vec{r}')\,e^{-i\omega t}\ \frac{e^{i\omega|\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}$$

doing $\int dt'$ by using the δ-function

$$\vec{A}_\omega(\vec{r}) = \frac{1}{c}\int d^3r'\ \vec{j}_\omega(\vec{r}')\ \frac{e^{i\omega|\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}$$

Assume source is localized, ie $\vec{j}_\omega(\vec{r}) \approx 0$ for $|\vec{r}| > d$



Approx ①

for $r \gg d$, far from sources

$$\begin{aligned}|\vec{r}-\vec{r}'| &= \sqrt{r^2+r'^2-2rr'\cos\theta} \\ &= r\sqrt{1+\left(\frac{r'}{r}\right)^2-\frac{2r'}{r}\cos\theta} \\ &\approx r\left(1-\frac{r'}{r}\cos\theta\right) \\ &\approx r-\vec{r}'\cdot\hat{r} + O\left(\frac{r'}{r}\right)^2\end{aligned}$$

$$\hat{r} \equiv \frac{\vec{r}}{r}$$

$$\vec{A}_\omega(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{J}_\omega(\vec{r}')\, e^{ik(r-\vec{r}'\cdot\hat{r})}}{r-\vec{r}'\cdot\hat{r}} \qquad \text{where } k \equiv \frac{\omega}{c}$$

$$= \frac{e^{ikr}}{cr}\int d^3r' \, \frac{\vec{J}_\omega(\vec{r}')\, e^{-ik\vec{r}'\cdot\hat{r}}}{1-\frac{\hat{r}\cdot\vec{r}'}{r}}$$

$$\approx \frac{e^{ikr}}{cr}\int d^3r' \, \vec{J}_\omega(\vec{r}')\, e^{-ik\hat{r}\cdot\vec{r}'}\left(1+\frac{\hat{r}\cdot\vec{r}'}{r}\right)$$

when combine with the $e^{-i\omega t}$ piece, this gives outgoing spherical wave $\frac{e^{i(kr-\omega t)}}{r}$

oscillating charge radiates outgoing spherical electromagnetic waves

the $\int d^3r' \, \vec{J}_\omega(\vec{r}') \ldots$ term will determine the angular dependence of the radiation.

Approx 2 $\quad \lambda \gg d$ long wave length approx

or $kd \ll 1 \Rightarrow \frac{\omega}{c} d \ll 1$ or $\frac{d}{\tau} \ll c$

where τ is period of oscillation.

Since $\frac{d}{\tau}$ is max speed of the oscillating charges $\Rightarrow \lambda \gg d$ is a non-relativistic approximation

$kd \ll 1 \Rightarrow e^{-ik\hat{r}\cdot\vec{r}'} \approx 1 - ik\hat{r}\cdot\vec{r}'$ + higher order

$$\vec{A}_\omega(\vec{r}) = \frac{e^{ikr}}{cr} \int d^3r' \, \vec{j}_\omega(\vec{r}') \, (1 - ik\hat{r}\cdot\vec{r}')\left(1 + \frac{\hat{r}\cdot\vec{r}'}{r}\right)$$

$$= \frac{e^{ikr}}{cr} \int d^3r' \, \vec{j}_\omega(\vec{r}') \left[1 + \hat{r}\cdot\vec{r}'\left(\frac{1}{r} - ik\right)\right]$$

+ higher order in $\frac{d}{r}$ or kd

$$\vec{A}_\omega(\vec{r}) = \frac{e^{ikr}}{r}\left[\vec{I}_1 + \left(\frac{1}{r} - ik\right)\vec{I}_2\right]$$

where $\vec{I}_1 \equiv \frac{1}{c}\int d^3r' \, \vec{j}_\omega(\vec{r}')$

$\vec{I}_2 \equiv \frac{1}{c}\int d^3r' \, \hat{r}\cdot\vec{r}' \, \vec{j}_\omega(\vec{r}')$

Consider first $\vec{I}_1$ ith component ($\vec{I}_1$ vanishes in statics)

$\int d^3r \, j_i(\vec{r}) = -\int d^3r \, r_i \, \vec{\nabla}\cdot\vec{j}$ integration by parts

$= \int d^3r \, r_i \frac{\partial \rho}{\partial t}$ as $\vec{\nabla}\cdot\vec{j} + \frac{\partial \rho}{\partial t} = 0$

$\int d^3r \, j_i(\vec{r}) = -i\omega \int d^3r \, r_i \, \rho_\omega(\vec{r})$

$\Rightarrow \vec{I}_1 = -\frac{i\omega}{c}\int d^3r \, \vec{r}\rho_\omega(\vec{r}) = -\frac{i\omega}{c}\vec{p}_\omega$

electric dipole moment