

For arbitrary charge distributions - not pure harmonic

For  $\vec{p}_\omega e^{-i\omega t}$  pure harmonic oscillation, we found the radiated fields in electric dipole approx are

$$\vec{E} = \vec{E}_\omega e^{-i\omega t}, \quad \vec{B} = \vec{B}_\omega e^{-i\omega t}$$

$$\vec{E}_\omega = -k^2 \frac{e^{ikr}}{r} \hat{r} \times (\hat{r} \times \vec{p}_\omega) = -\frac{\omega^2}{c^2} \frac{e^{i\omega r/c}}{r} \hat{r} \times (\hat{r} \times \vec{p}_\omega)$$

$$\vec{B}_\omega = k^2 \frac{e^{ikr}}{r} (\hat{r} \times \vec{p}_\omega) = \frac{\omega^2}{c^2} \frac{e^{i\omega r/c}}{r} (\hat{r} \times \vec{p}_\omega)$$

$$\text{as } k = \frac{\omega}{c}$$

For an arbitrarily time varying charge distribution with electric dipole moment

$$\vec{p}(t) = \int \frac{d\omega}{2\pi} \vec{p}_\omega e^{-i\omega t}$$

then solution for fields given by superposition

$$\vec{E}(\vec{r}, t) = \int \frac{d\omega}{2\pi} \vec{E}_\omega e^{-i\omega t}$$

$$= - \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-r/c)}}{r} \left( \frac{\omega^2}{c^2} \right) \hat{r} \times (\hat{r} \times \vec{p}_\omega)$$

$$= \frac{-1}{c^2 r} \hat{r} \times \left[ \hat{r} \times \int \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \vec{p}_\omega \omega^2 \right]$$

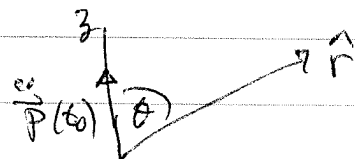
$$= \frac{1}{c^2 r} \hat{r} \times \left[ \hat{r} \times \frac{\partial^2}{\partial t^2} \int \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \vec{p}_\omega \right]$$

$$\boxed{\vec{E}(\vec{r}, t) = \frac{1}{c^2 r} \hat{r} \times \left[ \hat{r} \times \ddot{\vec{p}}(t - r/c) \right]} \quad \ddot{\vec{p}} = \frac{d^2 \vec{p}}{dt^2}$$

define  $t_0 \equiv t - r/c = \text{"retarded time"}$

in spherical coords, if  $\ddot{\vec{p}}(t_0)$  is along  $\hat{z}$

$$\vec{E}(\vec{r}, t) = \frac{\ddot{p}(t_0)}{c^2 r} \sin \theta \hat{\theta}$$



Similarly

$$\vec{B}(\vec{r}, t) = \int \frac{d\omega}{2\pi} \vec{B}_\omega e^{-i\omega t}$$

$$= \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-r/c)}}{r} \left( \frac{\omega^2}{c^2} \right) (\hat{r} \times \vec{p}_\omega)$$

$$= \frac{-1}{c^2 r} \hat{r} \times \frac{\partial^2}{\partial t^2} \int \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \vec{p}_\omega$$

$$\boxed{\vec{B}(\vec{r}, t) = \frac{-1}{c^2 r} \hat{r} \times \ddot{\vec{p}}(t_0)}$$

$$\vec{B}(\vec{r}, t) = \frac{\ddot{p}(t_0)}{c^2 r} \sin \theta \hat{\phi} \quad \text{in spherical coords}$$

Poynting vector

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = \frac{c}{4\pi} \left( \frac{1}{c^2 r} \right)^2 \left[ \ddot{p}(t_0) \right]^2 \sin^2 \theta \hat{r}$$

Total power radiated through a sphere of radius  $r$  is

$$\begin{aligned}
 P &= \oint da \hat{r} \cdot \vec{S} = 2\pi \int_0^\pi d\theta \sin\theta r^2 \hat{r} \cdot \vec{S} \\
 &= \frac{[\ddot{\vec{p}}(t_0)]^2}{2c^3} \underbrace{\int_0^\pi d\theta \sin^3\theta}_{4/3}
 \end{aligned}$$

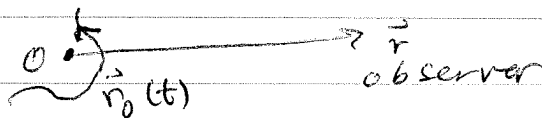
$$P = \frac{2}{3c^3} [\ddot{\vec{p}}(t_0)]^2$$

For a point charge moving along a trajectory  $\vec{r}_0(t)$

$$\vec{p}(t) = q \vec{r}_0(t)$$

$$\ddot{\vec{p}}(t) = q \ddot{\vec{r}}_0(t) = q \vec{a}(t)$$

↑ acceleration



$$P = \frac{2}{3} \frac{q^2 a^2(t_0)}{c^3}$$

Larmor's formula

← total power passing through a sphere of radius  $r$  at time  $t$  is due to acceleration at retarded time  $t_0 = t - r/c$

power radiated  $\propto (\text{acceleration})^2$

Larmor's formula above only holds in the non-relativistic limit since it is based on the electric dipole approx.

# Special Relativity

- 1) Speed of light is constant in all inertial frames of reference
- 2) Physical laws must look the same in all inertial frames of reference - there is no experiment that can determine the "absolute" velocity of any inertial frame

⇒ If a flash of light goes off at the origin of some coord system, the outgoing wavefronts look spherical in all inertial frames.  
Equation of wavefront is  $r^2 - c^2t^2 = 0$

⇒  $(x, y, z, t)$  coords in one inertial frame  $K$   
 $(x', y', z', t')$  coords in another inertial frame  $K'$  that moves with velocity  $\vec{v} = v\hat{x}$  with respect to  $K$ .

What is the transformation that relates coords in  $K'$  to coords in  $K$

$$y = y', \quad z = z'$$

(origins of  $K$  and  $K'$   
coincide when  $t = t' = 0$ )

$$\Rightarrow c^2t^2 - x^2 = c^2t'^2 - x'^2$$

$$\Rightarrow \frac{(ct+x)}{(ct'+x')} \frac{(ct-x)}{(ct'-x')} = 1$$

Expect transformation to be linear

$$\Rightarrow \begin{aligned} ct' + x' &= (ct+x) f \\ ct' - x' &= (ct-x) f^{-1} \end{aligned}$$

for some constant  $f$ . Write  $f = e^{-y}$   $y$  is rapidity

Solve for  $ct'$  and  $x'$  in terms of  $ct$  and  $x$

$$ct' = ct \left( \frac{e^y + e^{-y}}{2} \right) - x \left( \frac{e^y - e^{-y}}{2} \right)$$

$$x' = -ct \left( \frac{e^y - e^{-y}}{2} \right) + x \left( \frac{e^y + e^{-y}}{2} \right)$$

$$ct' = ct \cosh y - x \sinh y$$

$$x' = -ct \sinh y + x \cosh y$$

meaning of parameter  $y$

(at  $x=0$ )

the origin of  $K$  has trajectory  $x' = -vt'$  in  $K'$

$$\Rightarrow \frac{x'}{t'} = -v$$

from transformation above, with  $x=0$ , we get

$$\frac{x'}{ct'} = \frac{-ct \sinh y}{ct \cosh y} = -\tanh y$$

$$\text{so } \frac{v}{c} = \tanh y$$

$$\Rightarrow \cosh y = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \equiv \gamma$$

$$\sinh y = \left(\frac{v}{c}\right)\gamma$$

Lorentz Transformation

$$\begin{cases} ct' = \gamma ct - \gamma \left(\frac{v}{c}\right) x \\ x' = -\gamma \left(\frac{v}{c}\right) ct + \gamma x \end{cases}$$

Reverse transform obtained by taking  $v \rightarrow -v$  in above

$$\begin{cases} ct = \gamma ct' + \gamma \left(\frac{v}{c}\right) x' \\ x = \gamma \left(\frac{v}{c}\right) ct' + \gamma x' \end{cases}$$

### 4-vectors

4-position:  $X_\mu = (x_1, x_2, x_3, ict)$   $x_4 \equiv ict$   
 $X_\mu X_\mu \equiv \sum_{\mu=1}^4 X_\mu^2 = r^2 - c^2 t^2$  Lorentz invariant scalar  
 - has same value in all inertial frames

Lorentz transf is

$$\begin{cases} x_1' = \gamma \left( x_1 + i \left(\frac{v}{c}\right) x_4 \right) \\ x_2' = x_2 \\ x_3' = x_3 \\ x_4' = \gamma \left( x_4 - i \left(\frac{v}{c}\right) x_1 \right) \end{cases}$$

linear transf, can be represented by a matrix

or  $x_\mu' = a_{\mu\nu}(L) x_\nu$

$\hat{L}$  matrix of Lorentz transformation  $L$

$$a(L) = \begin{pmatrix} \gamma & 0 & 0 & i \frac{v}{c} \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i \frac{v}{c} \gamma & 0 & 0 & \gamma \end{pmatrix}$$

inverse:  $x_\mu = a_{\mu\nu}(L^{-1}) x_\nu'$

$a_{\mu\nu}(L^{-1})$  is given by taking  $v \rightarrow -v$  in  $a_{\mu\nu}(L)$

we see  $a_{\mu\nu}(L^{-1}) = a_{\nu\mu}(L)$

inverse = transpose

More generally

Since  $x_\mu^2$  is Lorentz invariant scalar,

$$x_\mu'^2 = a_{\mu\nu}(L) a_{\mu\lambda}(L) x_\nu x_\lambda = x_\lambda^2$$

$$\Rightarrow a_{\mu\nu}(L) a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow a_{\nu\mu}^t(L) a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow a_{\nu\mu}^t = a_{\mu\nu}^{-1}(L) \quad \text{transpose} = \text{inverse}$$

$a_{\mu\nu}$  is 4x4 orthogonal matrix

If  $L_1$  is a Lorentz transf from  $K$  to  $K'$

$L_2$  is a Lorentz transf from  $K'$  to  $K''$

Then the Lorentz transf from  $K$  to  $K''$  is given by the matrix

$$a(L_2 L_1) = a(L_2) a(L_1)$$

if  $L_1 = L$  and  $L_2 = L^{-1}$  so  $L_2 L_1 = I$  identity

$$\Rightarrow a^{-1}(L) = a(L^{-1})$$

$$dx_\mu = (dx_1, dx_2, dx_3, icdt)$$

$$-(dx_\mu)^2 \equiv c^2 ds^2 = c^2 dt^2 - dr^2 \quad \text{Lorentz invariant scalar}$$

$$ds^2 = dt^2 \left[ 1 - \frac{1}{c^2} \left( \frac{dx_1}{dt} \right)^2 - \frac{1}{c^2} \left( \frac{dx_2}{dt} \right)^2 - \frac{1}{c^2} \left( \frac{dx_3}{dt} \right)^2 \right]$$

$$ds^2 = \frac{dt^2}{\gamma^2}$$

$$\boxed{ds = \frac{dt}{\gamma}} \quad \text{proper time interval}$$