A 4-vector is any 4 numbers that transform under a Lorentz transformation the same way as does x_μ

<u>4-velocity</u> $\quad u_\mu \equiv \dfrac{dx_\mu}{ds} \equiv \dot{x}_\mu$

$$= \gamma \frac{dx_\mu}{dt}$$

space components $\vec{u} = \gamma \vec{v}$

$u_4 = ic\gamma$

$$u_\mu u_\mu = \gamma^2 v^2 - c^2\gamma^2 = \gamma^2 (v^2 - c^2)$$

$$= \frac{v^2 - c^2}{1 - \frac{v^2}{c^2}} = -c^2$$

<u>4-acceleration</u> $\quad a_\mu \equiv \dfrac{du_\mu}{ds} = \gamma \dfrac{du_\mu}{dt}$

<u>4-gradient</u> $\quad \dfrac{\partial}{\partial x_\mu} \equiv \left(\vec{\nabla}, \dfrac{-i}{c}\dfrac{\partial}{\partial t} \right)$

proof $\dfrac{\partial}{\partial x_\mu}$ is a 4-vector

$$\frac{\partial}{\partial x_\mu'} = \frac{\partial x_\lambda}{\partial x_\mu'} \frac{\partial}{\partial x_\lambda} \qquad \text{but } \frac{\partial x_\lambda}{\partial x_\mu'} = a_{\lambda\mu}(L^{-1}) = a_{\mu\lambda}(L)$$

$$= a_{\mu\lambda}(L) \frac{\partial}{\partial x_\lambda}$$

so transforms same as x_μ

$$\left(\frac{\partial}{\partial x_\mu} \right)^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \qquad \text{wave equation operator!}$$

<u>inner products</u>

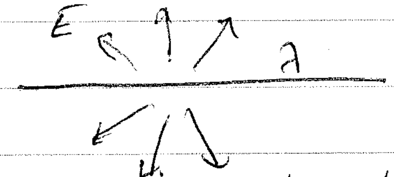
If u_μ and v_μ are 4-vectors, then $u_\mu v_\mu$ is Lorentz invariant scalar

Electromagnetism

Clearly $\vec{E}$ + $\vec{B}$ must transform into each other under Lorentz transf.

in inertial frame K

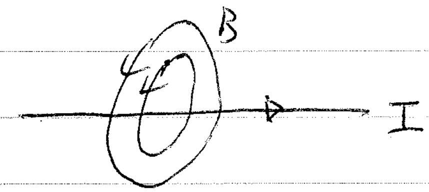
stationary line charge λ



cylindrical outward electric field

no B-field

in frame K' moving with $\vec{v}$ ∥ to wire



moving line charge gives current
⇒ B circulating around wire
as well as outward radial E

Lorentz force

$$\vec{F} = q\vec{E} + q\frac{\vec{v}}{c}\times\vec{B}$$

What is the velocity $\vec{v}$ here? velocity with respect to what inertial frame? Clearly $\vec{E}$ and $\vec{B}$ must change from one inertial frame to another if this force law can make sense.

Charge density

Consider charge ΔQ contained in a vol ΔV.
ΔQ is a Lorentz invariant scalar.

Consider the reference frame in which the charge is instantaneously at rest. In this frame

$\Delta Q = \overset{\circ}{\rho}\, \Delta \overset{\circ}{V}$ $\qquad$ $\overset{\circ}{\rho}$ is charge density in the rest frame

$\Delta \overset{\circ}{V}$ is volume in the rest frame

$\overset{\circ}{\rho}$ is Lorentz invariant by definition

Now transform to another frame moving with $\vec{v}$ with respect to rest frame

ΔQ remains the same

$\Delta V = \dfrac{\Delta \overset{\circ}{V}}{\gamma}$ $\qquad$ volume contracts in direction ∥ to $\vec{v}$

$$\rho = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q}{\Delta \overset{\circ}{V}}\gamma = \overset{\circ}{\rho}\,\gamma$$

Current density is $\vec{j} = \rho\vec{v} = \gamma\vec{v}\cdot\dfrac{\rho}{\gamma} = \overset{\circ}{\rho}\,\vec{u}$

Define <u>4-current</u> $\boxed{j_\mu = (\vec{j}, ic\rho)} = \overset{\circ}{\rho}(\vec{u}, ic\gamma)$

$= \overset{\circ}{\rho}\, u_\mu$

It is 4-vector since u_μ is 4-vector and $\overset{\circ}{\rho}$ is Lorentz invariant scalar.

charge conservation

$$\vec{\nabla}\cdot\vec{j} + \frac{\partial \rho}{\partial t} = \boxed{\frac{\partial j_\mu}{\partial x_\mu} = 0}$$

Equation for potentials in Lorentz gauge

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\frac{4\pi}{c}\vec{J}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -4\pi\rho$$

$\dfrac{\partial^2}{\partial x_\mu^2} = \left(\nabla^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\right)$ is Lorentz invariant operator

<u>4-potential</u> $\boxed{A_\mu = (\vec{A},\ i\phi)}$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A_\mu = \boxed{-\frac{4\pi}{c}J_\mu = \frac{\partial^2 A_\mu}{\partial x_\lambda^2}}$$

Lorentz gauge condition is

$$\vec{\nabla}\cdot\vec{A} + \frac{\partial\phi}{c\,\partial t} = \frac{\partial A_\mu}{\partial x_\mu} = 0$$

<u>Electric and magnetic fields</u>

$$B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \qquad i, j, k \text{ cyclic permutation of } 1, 2, 3$$

$$E_i = -\frac{\partial\phi}{\partial x_i} - \frac{\partial A_i}{c\,\partial t} = i\left(\frac{\partial A_4}{\partial x_i} - \frac{\partial A_i}{\partial x_4}\right)$$

Define <u>field stress tensor</u>

$$\boxed{F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = -F_{\nu\mu}}$$

$$= \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix}$$

"curl" of a 4-vector is a 4×4 anti symmetric 2nd rank tensor

Inhomogeneous Maxwell's equations can be written in the form

$$\boxed{\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} j_\mu} \Rightarrow \left[\begin{array}{l} \vec{\nabla}\cdot\vec{E} = 4\pi\rho \\ \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j} \end{array}\right]$$

$$= \frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) = \frac{\partial}{\partial x_\mu}\underbrace{\left(\frac{\partial A_\nu}{\partial x_\nu}\right)}_{=0} - \frac{\partial^2 A_\mu}{\partial x_\nu^2}$$

$$\Rightarrow -\frac{\partial^2 A_\mu}{\partial x_\nu^2} = \frac{4\pi}{c} j_\mu$$ agrees with previous equation for A_μ

transformation law for 2nd rank tensor $F_{\mu\nu}$

$$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu}$$ use $A'_\mu = a_{\mu\sigma} A_\sigma$, $\frac{\partial}{\partial x'_\mu} = a_{\mu\lambda}\frac{\partial}{\partial x_\lambda}$

$$= a_{\nu\lambda} a_{\mu\sigma} \frac{\partial A_\lambda}{\partial x_\sigma}$$

$$- a_{\mu\sigma} a_{\nu\lambda} \frac{\partial A_\sigma}{\partial x_\lambda}$$

$$F'_{\mu\nu} = a_{\mu\sigma} a_{\nu\lambda} F_{\sigma\lambda}$$ lets one find $\vec{E}'$ and $\vec{B}'$ if one knows $\vec{E}$ and $\vec{B}$

For n^{th} rank tensor

$$T'_{\mu_1\mu_2\cdots\mu_n} = a_{\mu_1\nu_1} a_{\mu_2\nu_2} \cdots a_{\mu_n\nu_n} T_{\nu_1\nu_2\cdots\nu_n}$$

$\frac{\partial F_{\mu\nu}}{\partial x_\nu}$ is a 4-vector: proof:

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\sigma} a_{\nu\lambda} a_{\nu\gamma} \frac{\partial F_{\sigma\lambda}}{\partial x_\gamma}$$

but $a_{\nu\lambda} = a^{-1}_{\lambda\nu}$ since inverse = transpose

$a_{\nu\lambda} a_{\nu\gamma} = a^{-1}_{\lambda\nu} a_{\nu\gamma} = \delta_{\lambda\gamma}$

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\sigma} \frac{\partial F_{\sigma\lambda}}{\partial x_\gamma} \delta_{\lambda\gamma} = a_{\mu\sigma} \frac{\partial F_{\sigma\lambda}}{\partial x_\lambda}$$ transforms like 4-vector

To write the <u>homogeneous Maxwell Equations</u>

Construct 3rd rank co-variant tensor

$$\boxed{G_{\mu\nu\lambda} = \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu}}$$

transforms as $G'_{\mu\nu\lambda} = a_{\mu\alpha} a_{\nu\beta} a_{\lambda\gamma} G_{\alpha\beta\gamma}$

in principle G has $4^3 = 64$ components

But can show that G is antisymmetric in exchange of any two indices

$$G_{\nu\mu\lambda} = \frac{\partial F_{\nu\mu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\nu}}{\partial x_\mu} + \frac{\partial F_{\mu\lambda}}{\partial x_\nu}$$

$$= -\frac{\partial F_{\mu\nu}}{\partial x_\lambda} - \frac{\partial F_{\nu\lambda}}{\partial x_\mu} - \frac{\partial F_{\lambda\mu}}{\partial x_\nu}$$ as F antisymmetric

$$= -G_{\mu\nu\lambda}$$

Also $G_{\mu\nu\lambda} = 0$ if any two indices are equal

$\Rightarrow$ only 4 independent components

G_{012}, G_{013}, G_{023}, G_{123}

all other components either vanish or are $\pm$ one of the above.

The 4 homogeneous Maxwell Equations:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad , \quad \vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

can be written as

$$\boxed{G_{\mu\nu\lambda} = 0}$$

to see, substitute in definition of G the definition of F

$$G_{\mu\nu\lambda} = \frac{\partial^2 A_\nu}{\partial x_\lambda \partial x_\mu} - \frac{\partial^2 A_\mu}{\partial x_\lambda \partial x_\nu} + \frac{\partial^2 A_\mu}{\partial x_\nu \partial x_\lambda} - \frac{\partial^2 A_\lambda}{\partial x_\nu \partial x_\mu} + \frac{\partial^2 A_\lambda}{\partial x_\mu \partial x_\nu} - \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\lambda}$$

all terms cancel in pairs

$= 0$

$$G_{123} = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$G_{012} = -i \left[\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{c\,\partial t} \right]_z = 0$$

z component Faraday's law

Another way to write homogeneous Maxwell Equations

Define $\epsilon_{\mu\nu\lambda\sigma} = \begin{cases} +1 & \text{if } \mu\nu\lambda\sigma \text{ is even permutation of } 1234 \\ -1 & \text{if } \mu\nu\lambda\sigma \text{ is odd permutation of } 1234 \\ 0 & \text{otherwise} \end{cases}$

4-d LeviCivita symbol

Define

$$\tilde{F}_{\mu\nu} = \frac{1}{2i} \epsilon_{\mu\nu\lambda\sigma} F_{\lambda\sigma}$$

pseudo-tensor

has wrong sign under parity transf

$$= \begin{pmatrix} 0 & -E_3 & E_2 & -iB_1 \\ E_3 & 0 & -E_1 & -iB_2 \\ -E_2 & E_1 & 0 & -iB_3 \\ iB_1 & iB_2 & iB_3 & 0 \end{pmatrix}$$

$\frac{\partial \tilde{F}_{\mu\nu}}{\partial x_\nu} = 0$ gives homogeneous Maxwell equations

$$\left.\begin{array}{l} \frac{1}{2} F_{\mu\nu} F_{\mu\nu} = B^2 - E^2 \\ -\frac{1}{4} F_{\mu\nu} \tilde{F}_{\mu\nu} = \vec{B}\cdot\vec{E} \end{array}\right\} \text{Lorentz invariant scalars}$$