Green's function - part II

Green's 2nd identity

\[ \int d^3r' \left( \phi \nabla'^2 - 4 \nabla'^2 \phi \right) = \oint_{\partial S} \left( \phi \frac{\partial \Phi'}{\partial m'} - 4 \frac{\partial \Phi'}{\partial m'} \right) \]

Apply above with \( \phi(r') \) electrostatic potential with \( \nabla'^2 \phi = -4\pi \rho(r') \)
\( \Phi(r') = G(r, r') \) the Green function satisfying

\( \nabla'^2 G(r, r') = -4\pi \delta(r-r') \)

We saw one solution of above is \( G(r, r') = \frac{1}{|r-r'|} \)

but a more general solution is

\[ G(r, r') = \frac{1}{|r-r'|} + F(r, r') \]

where \( \nabla'^2 F(r, r') = 0 \), for \( r' \) in volume \( V \)
we will choose \( F(r, r') \) to simplify solution of \( \phi \)

\[ \Rightarrow \int d^3r' \left( \phi(r') \nabla'^2 G(r, r') - G(r, r') \nabla'^2 \phi(r') \right) \]

\[ = \int d^3r' \left( \phi(r') \left[ -4\pi \delta(r-r') \right] - G(r, r') \left[ -4\pi \delta(r-r') \right] \right) \]

\[ = -4\pi \phi(r) + 4\pi \int d^3r' G(r, r') \rho(r') \]

\[ = \oint_{\partial S} \left( \phi \frac{\partial \Phi'}{\partial m'} - G \frac{\partial \Phi'}{\partial m'} \right) \]
\[ \phi(\vec{r}) = \int_V d^3r' \; G(\vec{r}, \vec{r}') \; \rho(\vec{r}') + \oint_S \frac{\partial \phi(\vec{r}')}{\partial n'} \; G(\vec{r}, \vec{r}') \; \partial G(\vec{r}, \vec{r}') \]

Consider the **Dirichlet boundary problem**. If we can choose \( F(\vec{r}, \vec{r}') \) such that \( G_B(\vec{r}, \vec{r}') = 0 \) for \( \vec{r}' \) on the boundary surface \( S \), then above reduces to

\[
\begin{bmatrix}
\phi(\vec{r}) = \int_V d^3r' \; G_B(\vec{r}, \vec{r}') \; \rho(\vec{r}') - \oint_S \frac{\partial \phi(\vec{r}')}{\partial n'} \; \partial G_B(\vec{r}, \vec{r}')
\end{bmatrix}
\]

Since \( \rho(\vec{r}) \) is specified in \( V \), and \( \phi(\vec{r}) \) is specified on \( S \), above then gives desired solution for \( \phi(\vec{r}) \) inside volume \( V \).

Finally \( G_B \) is therefore equivalent to finding an \( F(\vec{r}, \vec{r}') \) such that \( \nabla^2 F(\vec{r}, \vec{r}') = 0 \) for \( \vec{r}' \) in \( V \) (solves Laplace eqn) and

\[ F(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} \]

for \( \vec{r}' \) on boundary surface \( S \)

Always exists unique solution for \( F \).
Next consider Neumann boundary problem.

One might think to find \( F(r, \mathbf{r}') \) such that \( \frac{\partial G(r, \mathbf{r}')}{\partial m'} = 0 \) on boundary surface. But this is not possible.

Consider \( \int V \nabla^2 G(r, \mathbf{r}') \, d^3 r' = \int V' \cdot \nabla G(r, \mathbf{r}') \, d^3 r' \)

\[ = \int_S \mathbf{\nabla'} G(r, \mathbf{r}') \cdot \hat{m} \, da' \]

\[ = \int_S \frac{\partial G(r, \mathbf{r}')}{\partial m'} \, da' = -4\pi \text{ since } \nabla^2 G = -4\pi \delta(r-r') \]

So we can't have \( \frac{\partial G}{\partial m'} = 0 \) for \( r' \) on \( S' \).

Simplest choice is then \( \frac{\partial G_N(r, \mathbf{r}')}{\partial m'} = -4\pi \text{ for } r' \text{ on } S \)

Then

\[ \phi(\mathbf{r}) = \int V d^3 r' G_N(r, \mathbf{r}') \, f(\mathbf{r}') + \int \frac{da'}{4\pi} G_N(r, \mathbf{r}') \frac{\partial \phi(r)}{\partial m'} \]

\[ + \int \frac{da'}{4\pi} \phi(r') \left( -\frac{4\pi}{S'} \right) \]

\[ \left[ \phi(\mathbf{r}') = \int V d^3 r'' G_N(\mathbf{r'}, \mathbf{r}') \, f(\mathbf{r}') + \int \frac{da'}{4\pi} G_N(\mathbf{r'}, \mathbf{r}') \frac{\partial \phi(r')}{\partial m'} \right] \]

\[ + \left< \phi \right>_S \]

Since \( f(r) \) is specified in \( V \)

and \( \frac{\partial \phi}{\partial m} \) is specified on \( S' \)

\[ \text{constant = average value of } \phi \text{ on surface } S' \]

Above gives solution \( \phi(r) \) in \( V \) within additive constant \( \left< \phi \right>_S \)

\[ \text{Since } \nabla^2 \phi = 0 \] the const. \( \left< \phi \right>_S \) is of no consequence.
Finding \( G_N(\vec{r}, \vec{r}') \) is therefore equivalent to finding another \( F(\vec{r}, \vec{r}') \) such that

\[
\nabla'^2 F(\vec{r}, \vec{r}') = 0 \quad \text{for} \quad \vec{r}' \in V
\]

and

\[
\frac{\partial F(\vec{r}, \vec{r}')}{\partial \vec{r}'_i} = -\frac{4\pi}{\Omega} \quad \text{for} \quad \vec{r}' \text{ on surface } S'
\]

always exists a unique solution (within additive constant).

While \( G_D \) and \( G_N \) always exist in principle, they depend in detail on the shape of the surface \( S \) and are difficult to find except for simple geometries.

In proceeding we defined \( G \) by

\[
\nabla'^2 G(\vec{r}, \vec{r}') = -\frac{4\pi\delta(\vec{r} - \vec{r}')}{\Omega}
\]

But our earlier interpretation of \( G(\vec{r}, \vec{r}') \) was that it was potential at \( \vec{r} \) due to point source at \( \vec{r}' \), i.e.

\[
\nabla^2 G(\vec{r}, \vec{r}') = -\frac{4\pi}{\Omega} \delta(\vec{r} - \vec{r}') \quad \text{. Note, for general surface } S' \text{, } G(\vec{r}, \vec{r}') \text{ is not in general a function of } \vec{r} - \vec{r}' \text{ but depends on } \vec{r} \text{ and } \vec{r}' \text{ separately. But the equivalence of the two definitions of } G \text{ above is obtained by noting that one can prove the symmetry property}
\]

\[
G(\vec{r}, \vec{r}') = G(\vec{r}', \vec{r})
\]

for Dirichlet b.c., and one can impose it as an additional requirement for Neumann b.c. (see Jackson, end section 1.10).