Some more problems

Infinite conducting wire of radius $R$ with line charge density $\lambda = \text{charge per unit length}$

Surface charge $\sigma = \frac{\lambda}{2\pi R}$

Expect cylindrical symmetry $\Rightarrow \phi$ depends only on cylindrical coord $r$.

$\nabla^2 \phi = 0 \quad \text{for} \quad r > R, \quad r < R$

Use $\nabla^2$ in cylindrical coords — only radial term non-vanishing

$\nabla^2 \phi = \frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) = 0$

\[
\begin{align*}
  r \frac{d\phi}{dr} &= C_0 \quad \text{constant} \\
  \frac{d\phi}{dr} &= \frac{C_0}{r} \\
  \phi(r) &= C_0 \ln r + C_1 \quad \text{const}
\end{align*}
\]

Note: One cannot now choose $\phi \to 0$ as $r \to \infty$.

One needs to fix zero of $\phi$ at some other radius. A convenient choice is $r = R$, but any other choice could also be made.
\[ \phi_{\text{out}} = C_0 \ln r + C_1 \]
\[ \phi_{\text{in}} = C_0 \ln r + C_1 \]

\[ \phi_{\text{in}} = \text{const in conductor} \Rightarrow C_0 \ln r = 0 \]
or \[ \phi_{\text{in}} \text{ should not diverge as } r \to 0 \Rightarrow C_0 \ln r = 0 \]

So \[ \phi_{\text{in}} = C_1 \text{ constant} \]

Boundary condition at \( r = R \)
\[ \left[ -\frac{d\phi_{\text{out}}}{dr} + \frac{d\phi_{\text{in}}}{dr} \right]_{r = R} = 4\pi \sigma \]

\[ \Rightarrow -\frac{C_0}{R} = 4\pi \sigma = 4\pi \left( \frac{\lambda}{2\pi R} \right) = \frac{2\lambda}{R} \]

\[ C_0 = -2\lambda \]

\[ \phi_{\text{out}} = -2\lambda \ln r + C_1 \]
continuity of \( \phi \)

\[ \phi_{\text{in}} (R) = \phi_{\text{out}} (R) \Rightarrow C_1 \ln R = -2\lambda \ln R + C_1 \]

Remaining const \( C_1 \) is not too important as it is just a common additive constant to both \( \phi_{\text{in}} \) and \( \phi_{\text{out}} \) does not change \( E' = -\nabla \Phi \).

If use the condition \( \phi (R) = 0 \) then we can solve for \( C_1 \)
\[ 0 = -2\lambda \ln R + c_1^{\text{out}} \quad \Rightarrow \quad c_1^{\text{out}} = 2\lambda \ln R \]

\[ \Rightarrow \phi(r) = \begin{cases} -2\lambda \ln (r/R) & r > R \\ 0 & r < R \end{cases} \]

\[ \frac{\delta}{\text{infinite conducting half space}} \]

\[ \vec{E}(r) = \begin{cases} \frac{2\lambda}{r} & r > R \\ \vec{0} & r < R \end{cases} \]

\[ \sigma \text{ uniform surface charge density} \]

Expect \( \phi \) depends only on \( x \)

\[ \nabla^2 \phi = \frac{d^2 \phi}{dx^2} = 0 \]

\[ \Rightarrow \begin{cases} \phi'(x) = c_0' x + c_1^\prime & x > 0 \\ \phi'(x) = c_0^- x + c_1^- & x < 0 \end{cases} \]

For \( x < 0 \), \( \phi = \text{const on conductor} \quad \Rightarrow c_1^- = 0 \)

At \( x = 0 \), \( \phi \) continuous \quad \Rightarrow \phi'(0) = \phi'(0)^+ \quad \Rightarrow c_0^- = c_1^+ \)

\[ \frac{d \phi}{dx} \text{ discontinuous} \quad \Rightarrow \]

\[ -\frac{d \phi}{dx} \bigg|_{x=0} = 4\pi \sigma \]

\[ c_0^- = -4\pi \sigma \]

\[ \Rightarrow \phi(x) = \begin{cases} -4\pi \sigma x + c_1^+ & x > 0 \\ c_1^+ & x < 0 \end{cases} \]

\[ \text{const } c_1^+ \text{ does not change value of } \vec{E} \]
as for the wire, we cannot choose $\phi \to 0$ as $x \to 0$.

On set $\Phi = 0$ not

\[ -\nabla^2 \Phi = \vec{E} = \begin{cases} 4\pi \sigma \hat{x} & x > 0 \\ 0 & x < 0 \end{cases} \]

**Infinite charged plane**

Similar to previous problem, but now no conductor at $x < 0$, just free space on both sides of the charged plane at $x = 0$.

\[ \nabla^2 \phi = \frac{d^2 \phi}{dx^2} = 0 \implies \phi^+ = c_0^+ x + c_1^+ \quad x > 0 \\
\phi^- = c_0^- x + c_1^- \quad x < 0 \]

**Continuity of $\phi$ at $x = 0$**

\[ \phi^+(0) = \phi^-(0) \implies c_1^+ = c_1^- \]

**Discontinuity of $\frac{d\phi}{dx}$ at $x = 0$**

\[ -\frac{d\phi^+}{dx} + \frac{d\phi^-}{dx} = 4\pi \sigma \]

\[ -c_0^+ + c_0^- = 4\pi \sigma \]

Define $\varepsilon_0 = \frac{c_0^+ + c_0^-}{2}$

Then we can write
\[ C_0^< = \overline{C}_0 + 2\pi \sigma \]
\[ C_0^> = \overline{C}_0 - 2\pi \sigma \]

\[ \phi = \begin{cases} 
-2\pi \sigma x + \overline{C}_0 x + C_i^+ & x > 0 \\
2\pi \sigma x + \overline{C}_0 x + C_i^- & x < 0 
\end{cases} \]

\[ \frac{d\phi}{dx} \equiv \vec{E} = \begin{cases} 
(2\pi \sigma - \overline{C}_0) \hat{x} & x > 0 \\
-2\pi \sigma - \overline{C}_0 \hat{x} & x < 0 
\end{cases} \]

Const. \( C_i^\pm \) does not affect \( \vec{E} \) — additive const to \( \phi \)
\( \overline{C}_0 \) represents const uniform electric field \(-\overline{C}_0 \hat{x}\),
that exists independent of the charged surface

If we assumed that all \( \vec{E} \) fields are just those arising from the plane, then we can set \( \overline{C}_0 = 0 \).
Equivalently, if the plane is the only source of \( \vec{E} \),
then we expect \( \phi \) depends only on \(|x|\) by symmetry.
\[ \Rightarrow \quad C_0^< = -C_0^> \quad \text{and again} \quad \overline{C}_0 = 0. \]
In this case

\[ \phi(x) = \begin{cases} 
-2\pi \sigma x & x > 0 \\
2\pi \sigma x & x < 0 
\end{cases} \]

\[ \vec{E}(x) = \begin{cases} 
2\pi \sigma \hat{x} & x > 0 \\
-2\pi \sigma \hat{x} & x < 0 
\end{cases} \]

\( \vec{E} \) is constant but oppositely directed on either side of the charged plane.
Green's Theorem, Uniqueness, Green's function part II

We want to show that the boundary value problem we described is well posed — if there is a unique solution. We start by deriving Green's Theorem.

Consider \( \int \int_S d^3r \cdot \nabla \cdot \vec{A} = \oint_C d\vec{l} \cdot \vec{A} \) Gauss theorem

let \( \vec{A} = \phi \vec{\nabla} \psi \)

\( \phi \vec{\nabla} \psi \cdot \hat{m} = \frac{\partial \psi}{\partial m} \)

\( \Rightarrow \nabla \cdot \vec{A} = \phi \nabla^2 \psi + \vec{\nabla} \phi \cdot \vec{\nabla} \psi \)

\[ \triangle \]

\( \Rightarrow \int \int_S d^3r \left( \phi \nabla^2 \psi + \vec{\nabla} \phi \cdot \vec{\nabla} \psi \right) = \oint_C d\vec{l} \phi \frac{\partial \psi}{\partial m} \} \text{ Green's 1st identity} \)

let \( \phi \leftrightarrow \psi \)

\[ \triangle \]

\[ \Rightarrow \int \int_S d^3r \left( \psi \nabla^2 \phi + \vec{\nabla} \psi \cdot \vec{\nabla} \psi \right) = \oint_C d\vec{l} \psi \frac{\partial \phi}{\partial m} \}

\[ \text{Greens 2nd identity} \]

Apply Green's 2nd identity with \( \psi = \frac{1}{r-r'} \)

\( \hat{r}' \) is integration variable \( \phi \) is the scalar potential

with \( \nabla^2 \psi = -4\pi \rho \). Use \( \nabla^2 \psi = \nabla' \cdot \nabla' \psi = -4\pi \delta(r-r') \)

\[ \int d^3r' \left[ \phi(r') \left[ -4\pi \delta(r-r') \right] \right] - \left( \frac{1}{(r-r')} \right) \left( -4\pi \rho(r) \right) \]

\[ = \int d\vec{m}' \left[ \phi \frac{\partial \psi}{\partial m'} \left( \frac{1}{r-r'} \right) - \frac{1}{r-r'} \frac{\partial \phi}{\partial m'} \right] \]
If \( \mathbf{r} \) lies within the volume \( V \), then

\[
\phi(\mathbf{r}) = \frac{1}{V} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \oint d\mathbf{a}' \left[ \frac{1}{4\pi} \frac{\partial \phi}{\partial n'} - \phi \frac{2}{3m'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right]
\]

\( \ast \ast \)

Note: if \( \mathbf{r} \) lies outside the volume \( V \), then

\[
0 = \frac{1}{V} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \oint d\mathbf{a}' \left[ \frac{1}{4\pi} \frac{\partial \phi}{\partial n'} - \phi \frac{2}{3m'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right]
\]

\( \ast \ast \ast \)

Potential from a surface charge density

\[
\sigma = \frac{1}{4\pi} \frac{\partial \phi}{\partial n'}
\]

Potential from a dipole strength density

\[
\frac{\phi}{4\pi}
\]

From (\( \ast \ast \)), if \( S \to \infty \) and \( \mathbf{E} \sim \frac{2\phi}{m} \to 0 \) faster than \( \frac{1}{r} \), then the surface integral vanishes and we recover Coulomb's law

\[
\phi(\mathbf{r}) = \frac{1}{V} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}
\]

(\( \ast \)) gives the generalization of Coulomb's law to a system with a finite boundary.

For a charge-free volume \( V \), i.e. \( \rho(\mathbf{r}) = 0 \) in \( V \), the potential everywhere is determined by the potential and its normal derivative on the surface.

But one cannot in general freely specify both \( \phi \) and \( \frac{\partial \phi}{\partial n} \) on the boundary surface since the resulting \( \phi \) from (\( \ast \)) would not in general obey Laplace's equation \( \nabla^2 \phi = 0 \), nor would (\( \ast \ast \ast \)) vanish.
Specifying both $\phi$ and $\frac{\partial \phi}{\partial n}$ on surfaces is known as "Cauchy" boundary conditions — for Laplace's equation, Cauchy b.c. overspecify the problem and a solution cannot in general be found.

**Uniqueness**

If we have a system of charges in vol $V$, and neither the potential $\phi$, nor its normal derivative $\frac{\partial \phi}{\partial n}$, is specified on the surfaces of $V$, then there is a unique solution to Poisson's equation inside $V$. Specifying $\phi$ is known as Dirichlet boundary conditions. Specifying $\frac{\partial \phi}{\partial n}$ is known as Neumann boundary conditions.

**Proof:** Suppose we had two solutions $\phi_1$ and $\phi_2$, both with $-V^2 \phi = q\rho$ inside $V$, and obeying specified b.c. on surface of $V$.

Define $U = \phi_2 - \phi_1 \Rightarrow V^2 U = 0$ inside $V$ and $U = 0$ on surface $S$ — for Dirichlet b.c.

or $\frac{\partial U}{\partial n} = 0$ on surface $S$ — for Neumann b.c.

Use Green's 1st identity with $\phi = U = V$

$$\int_{V} \left( V^2 V + \nabla V \cdot \nabla V \right) \, dV = \oint_{S} V \frac{\partial V}{\partial n} \, dS$$

as $V^2 U = 0$ as $U \Rightarrow \frac{\partial U}{\partial n} = 0$
\[ \int d^3r \sqrt{\mathbf{u}} \mathbf{u}^2 = 0 \quad \Rightarrow \quad \nabla \mathbf{u} = 0 \quad \Rightarrow \quad \mathbf{u} = \text{const} \]

For Dirichlet b.c., \( U = 0 \) on surface \( S \), so \( \text{const} = 0 \) and \( \phi_1 = \phi_2 \). Solution is unique.

For Neumann b.c., \( \phi_1 \) and \( \phi_2 \) differ only by an arbitrary constant. Since \( \mathbf{E} = -\nabla \phi \), the electric fields \( \mathbf{E}_1 = -\nabla \phi_1 \) and \( \mathbf{E}_2 = -\nabla \phi_2 \) are the same.

Additionally, if boundary surface \( S \) consists of several disjoint pieces, then solution is unique if specify \( \phi \) on some pieces and \( \frac{\partial \phi}{\partial n} \) on other pieces.

Solution of Poisson's equation with both \( \phi \) and \( \frac{\partial \phi}{\partial n} \) specified on the same surface \( S \) (Cauchy b.c.) does not in general exist, since specifying either \( \phi \) or \( \frac{\partial \phi}{\partial n} \) alone is enough to give a unique solution.
Image Charge Method

For simple geometries, can try to obtain \( G_D \) or \( G_W \) by placing a set of "image charges" outside the volume of interest \( V \), i.e., on the "other side" of the system boundary surface \( S \). Because these image charges are outside \( V \), they contribute to the potential inside \( V \) obeying \( \nabla^2 \phi^\text{image} = 0 \), as necessary. Choose location of image charges so that total \( \phi \) has desired boundary condition.

1) Charge in front of infinite grounded plane

\[
\begin{align*}
\phi &= \frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{-q}{\sqrt{x^2 + y^2 + (z+d)^2}} \\
if \text{we find a solution to above, it is the unique solution}
\end{align*}
\]

Solution - put fictitious image charge \(-q\) at \( z = -d \)

\[ \phi = \text{Coulomb potential from the real charge} + \text{the image} \]

\[ \phi(r) = \frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{-q}{\sqrt{x^2 + y^2 + (z+d)^2}} \]

above satisfies \( \phi(x, y, 0) = 0 \) as required

Also, \( \nabla^2 \phi = -\frac{4\pi q}{\sqrt{(r-d)^2}} + \frac{4\pi q}{\sqrt{(r+d)^2}} \)

\[ = -\frac{4\pi q}{\sqrt{(r-d)^2}} \quad \text{for region } z > 0 \]