Can now find \( \hat{E} \) for \( z > 0 \):

\[
\hat{E} = -\vec{\nabla} \phi
\]

In particular:

\[
E_z = -\frac{\partial \phi}{\partial z} = +\frac{1}{2} \int \left( \frac{2(3-x)}{x^2+y^2+(3-x)^2} \right)^{3/2} - \left( \frac{2(3+x)}{x^2+y^2+(3+x)^2} \right)^{3/2}
\]

\[
E_z = \phi \left[ \frac{(3-x)}{x^2+y^2+(3-x)^2} \right]^{3/2} - \left[ \frac{(3+x)}{x^2+y^2+(3+x)^2} \right]^{3/2}
\]

We can use above to compute the surface charge density \( \sigma(x,y) \) induced on the surface of the conductor plane. At conductor surface:

\[
-\frac{\partial \phi}{\partial n} = 4\pi \sigma
\]

\[
\sigma = -\frac{1}{4\pi} \frac{2\phi}{\partial z} = \frac{1}{4\pi} E_z \quad (x,y,z=0)
\]

\[
\sigma(x,y) = \frac{\phi}{4\pi} \left[ \frac{-d}{(x^2+y^2+d^2)^{3/2}} - \frac{d}{(x^2+y^2+d^2)^{3/2}} \right]
\]

\[
= -\frac{\phi}{2\pi} \frac{d}{(x^2+y^2+d^2)^{3/2}} = \frac{-\phi d}{2\pi (r_1^2+d^2)^{3/2}}
\]

\( \sigma \)
Total induced charge is

\[ q_{\text{induced}} = \int_{-\infty}^{\infty} dx dy \sigma(x, y) \]

\[ = 2\pi \int_0^{\infty} dr_1 \sigma(r_1) \]

\[ = 2\pi \int_0^{\infty} dr_1 \frac{r_1 (-q d)}{2\pi \left(r_1^2 + d^2\right)^{3/2}} \]

\[ = -q d \left[ \frac{1}{r_1^{3/2}} \right]_0^{\infty} \]

\[ = -q d \left[ 0 - \frac{1}{d} \right] \]

\[ q_{\text{induced}} = -q \quad \text{induced charge = image charge} \]

Force on charge \( q \) in front of conducting plane is due to the induced \( \sigma \). The E fields of this \( \sigma \) is, for \( z > 0 \), the same as the E field of the image charge.

\[ \Rightarrow \vec{F} = -\frac{q^2}{(2d)^2} \hat{z} = -\frac{q^2}{4d^2} \hat{z} \quad \text{attractive} \]

Work done to move \( q \) into position from infinity is

\[ W = -\int_{-\infty}^{\infty} dx \cdot \vec{F} = -\int_{-\infty}^{\infty} dx F_z \]

\[ = -\frac{q^2}{4d^2} \]
\[ W = \int_{a}^{b} \left( -\frac{q^2}{4\pi\epsilon_0 z^2} \right) dx = -\frac{q^2}{4\pi\epsilon_0} \]

\[ W < 0 \Rightarrow \text{energy released} \]

\[ \text{Note: } W \text{ above is not the electrostatic energy that would be present if the image charge were real, i.e. it is not } \varphi \text{ at } (r=d/2) = -\frac{q^2}{2\pi \epsilon_0} \]

One way to see why is to note that as \( z \) is moved quasistatically in towards the conductor plane, the image charge also must be moving to stay equidistant on the opposite side.
2) point charge in front of a grounded \((\phi = 0)\) conducting sphere.

Place image charge \(q'\) inside sphere so that the combined \(\phi\) from \(q\) and \(q'\) vanishes on surface of sphere.

By symmetry, \(q'\) should lie on the same radial line as \(q\) does, call the distance \(q'\) from the origin “\(a\)

Potential at position \(\vec{r}\) is

\[
\phi(\vec{r}) = \frac{q}{|\vec{r} - s\hat{\hat{r}}|} + \frac{q'}{|\vec{r} - a\hat{z}|}
\]

\[
= \frac{q}{(\sqrt{r^2 + s^2 - 2sr\cos\theta})^{1/2}} + \frac{q'}{(r^2 + a^2 - 2ra\cos\theta)^{1/2}}
\]

Can we choose \(q'\) and \(a\) so that \(\phi(\vec{r}, \theta) = 0\) for all \(\theta\)?
\[ \phi(r_1, \theta) = \frac{q}{(r^2 + s^2 - 2sr \cos \theta)^{1/2}} + \frac{q'}{\left(\frac{s}{r} \right)^2 \left(\frac{s}{r} \right)^2 + \frac{a}{s} \left(\frac{s}{r} \right)^2 + \frac{s}{r} - 2sr \cos \theta \right)^{1/2} } \]

- Make denominators look alike

\[ r^2 + a^2 - 2ar \cos \theta = \frac{a}{s} \left( \frac{s}{r} \right)^2 + \frac{s}{r} - 2sr \cos \theta \]

- If choose \( \frac{s}{r} = R^2 \), ie \[ a = R^2 \frac{s}{r} \]

and then the denominator of the 2nd term is

\[ \left[ \frac{R^2}{s} \left( s^2 + R^2 - 2sr \cos \theta \right) \right]^{1/2} = \frac{R}{s} \left[ s^2 + R^2 - 2sr \cos \theta \right]^{1/2} \]

\[ \Rightarrow \phi(r_1, \theta) = \frac{q}{(r^2 + s^2 - 2sr \cos \theta)^{1/2}} + \frac{q' \left( \frac{s}{r} \right)}{\left( \frac{s}{r} \right)^2 + \frac{s}{r} - 2sr \cos \theta} \]

So choose \( q' \left( \frac{s}{r} \right) = -q \) \[ \Rightarrow \quad q' = -\frac{q}{r/s} \]

to get \( \phi(r_1, \theta) = 0 \)

Solution is

\[ \phi(r, \theta) = \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} - \frac{q \frac{R}{s}}{\left( \frac{R}{s} \right)^2 + \frac{s}{r} - 2sr \cos \theta} \]

\[ = \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} - \frac{q}{\left( \frac{s}{r} + \frac{R^2 - 2rs \cos \theta}{s^2} \right)^{1/2}} \]

Can get induced surface charge on sphere by

\[ 4\pi \sigma = \vec{E} \cdot \hat{n} = -\frac{\partial \phi}{\partial r} \bigg|_{r=R} \]

see Jackson Eq (2.5) for result
\[ \sigma(\theta) = -\frac{q}{4\pi RS} \frac{1}{(1+(R/s)^2} - 2(R/s) \cos \theta \right)^{3/2} \]

\( \sigma(\theta) \) is greatest at \( \theta = 0 \), as one should expect.

We integrate \( \sigma(\theta) \) to get total induced charge. One finds

\[ 2\pi \int_{0}^{\pi} \sin \theta R^2 \sigma(\theta) = q' = -\frac{q}{s} \]

In general, total induced charge = sum of all image charges

\textbf{Force of attraction of charge to sphere}

Force on \( q \) is due to electric field from induced charge \( \sigma \),

which is the same as the electric field from the image charge \( q' \).

\[ F = \frac{q' \hat{z}}{(s-a)^2} = -\frac{q^2 (R/s) \hat{z}}{(s - R^2 s)^2} = -\frac{q^2 R s \hat{z}}{(s^2 R^2)^2} \]

\textbf{Close to the surface of the sphere,} \( s \approx R \), so write \( s = R + d \)

where \( d \ll R \). Then

\[ F = \frac{-q^2 R s}{(s-R)^2(s+R)^2} = -\frac{q^2 R (R+d)}{d^2 (2R+d)^2} \approx -\frac{q^2}{4d^2} \]

get same result as for infinite flat grounded plane.

When \( q \) is so close to surface that \( d \ll R \), the charge does not "see" the curvature of the surface.
For from the surface, \( s \gg R \)

\[
F = \frac{g^2 g' \frac{\hat{z}}{z}}{(s-a)^2} = -\frac{g^2 R s}{(s^2 - R^2)^2} \cdot \frac{\hat{z}}{z} = -\frac{g^2 R}{s^3} \frac{\hat{z}}{z}
\]

\( F \sim \frac{1}{s^3} \) very different from flat plane also different from point charge.

Note: In proceeding two problems, what we found was a \( \Phi \) such that \( \nabla^2 \Phi = -4\pi \delta(\vec{r} - \vec{r}_0) \), for a charge at \( \vec{r}_0 \), and \( \Phi = 0 \) on the boundary. Such a \( \Phi \) is nothing more than \( G_0 \) the corresponding Green function for Dirichlet boundary conditions.

Suppose now that instead of a grounded sphere we have a sphere with fixed net charge \( Q \).

We want to add new image charge to represent this case. If we put \( g' = -\frac{Q}{2\pi \varepsilon_0 a} \) at \( a = R/s \) as before, the boundary condition of \( \Phi = \text{const} \) on surface \( r = R \) is met, but the net charge on the sphere is \( \Phi' \) (the induced charge) not the desired \( Q \). We therefore need to add new image charge(s) of total charge \( Q - Q' \) (so total image charge is \( Q' \)) in such a way that we keep \( \Phi \) constant on the surface of the sphere. The way to do this is to put \( Q - Q' \) at the origin!
Solution is

\[ \Phi (r, \theta) = \frac{q + q R/s}{R} + \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} - \frac{q}{(s^2 + R^2 - 2rs \cos \theta)^{1/2}} \]

The force on the charge \( q \) is due to the electric field of the images

\[ \mathbf{F} = \mathbf{F}^* = \frac{q (q + q R/s)}{s^2} + \frac{q f / \hat{z}}{(s - a)^2} \]

\[ F = \frac{q A}{s^2} + \frac{q^2 R/s}{s^2} - \frac{q^2 R/s}{(s - R^2/s)^2} \]

\[ = \frac{q A}{s^2} + \frac{q^2 R}{s^3} \left[ \frac{1}{s^3} - \frac{1}{s^3 (1 - R^2/s^2)^2} \right] \]

\[ = \frac{q A}{s^2} + \frac{q^2 R}{s^3} \left[ 1 - \frac{1}{(1 - R^2/s^2)^2} \right] \]

\[ F = \frac{q A}{s^2} - \frac{q^2 R^3}{s^5} \frac{2 - R^2}{(s^2 - R^2)^2} \]

For large \( s \gg R \) for from surface

\[ F \approx \frac{q A}{s^2} - \frac{q^2 R^3}{s^5} \]

leading term is just

\[ \text{Coulomb force between } q \text{ and } A \text{ at origin} \]

for \( A > 0 \), \( F \) is always repulsive for large enough \( s \).
For $s = R+d$, $d \ll R$ close to surface

\[ F = \frac{qA}{(R+d)^2} - \frac{q^2 R^3}{(R+d) (R^2 + d^2 + 2Rd - R^2)^2} \]

\[ \approx \frac{qA}{R^2} - \frac{q^2 R^3}{R} \frac{2-1}{4R^2 d^2} \]

\[ F \approx \frac{qA}{R^2} - \frac{q^2 R^3}{4d} \approx -\frac{q^2}{4d^2} \text{ for } d \text{ small enough} \]

$F$ is always attractive for small enough $d$, and is equal to the force in front of a grounded plane, no matter what is the value of $Q$. This is because the image charge $Q'$ lies so much closer to $q$ than does the $Q-q'$ at the origin, that it dominates the force.

The crossover from attractive to repulsive occurs at a distance $s$ that depends on $Q$. This distance is given by

\[ \frac{Q}{q} = \frac{R^3 s (2 - R^3 s^2)}{(s^2 - R^2)^2} = \left( \frac{R^3}{s} \right) \frac{2 - (R/s)^2}{\left[ 1 - (R/s)^2 \right]^2} \]

let $x = R/s \in (0,1)$

\[ \frac{Q}{q} = x^3 \frac{(2-x^2)}{(1-x^2)^2} \]

gives 5th order polynomial in $x$, no analytic solution, can solve graphically
For $\frac{Q}{q} = 1$, crossover is at $\frac{R}{s} = 0.62$

\[ S = 1.6R \]

$\frac{Q}{q} \approx 0.1$, crossover is at $\frac{R}{s} = 0.36$

\[ S = 2.8R \]

Distance $s$ at which a charge $q$ feels zero force when in front of a conducting sphere of radius $R$ with net charge $Q$. 

![Graph showing the relationship between $s/R$ and $Q/q$.]