Spherical Coordinates

\[ \Delta^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \phi^2} = 0 \]

\[ \phi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi) \]

\[ r^2 \Delta^2 \Phi = \frac{\partial^2 \Phi}{\partial r^2} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{R}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{R \Phi}{\sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 0 \]

\[ \frac{r^2 \sin^2 \theta}{\Phi} \Delta^2 \Phi = \frac{R}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial R \Phi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 R \Phi}{\partial \phi^2} = 0 \]

\[ \text{depends only on } r \text{ and } \theta \]

\[ \Phi = -\text{const} \]

\[ \text{depends only on } \phi = \text{const} \]

\[ \frac{1}{r} \frac{d^2 \Phi}{dr^2} = -m^2 \]

\[ \Rightarrow \Phi = e^{\pm im \phi} \quad m \text{ integer for } 2\pi \text{ periodicity in } \phi \]

\[ \Rightarrow \frac{R}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial R \Phi}{\partial \theta} \right) = m^2 \]

\[ \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{\partial R}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) - \frac{m^2}{\sin^2 \theta} = 0 \]

\[ \text{depends only on } r \]

\[ \text{depends only on } \theta \]

\[ \Phi = \text{const} \]

\[ \Phi = -\text{const} \]
Call the constant \( \ell (l+1) \)

\[
\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \ell (l+1) R = 0
\]

Solutions are of the form \( R(r) = a_2 r^l + b_2 r^{-l(l+1)} \)

Substitute in to verify

\[
\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \frac{d}{dr} \left( r^2 \left( l a_2 r^{l-1} - (l+1) b_2 r^{-l-2} \right) \right)
\]

\[
= \frac{d}{dr} \left( l a_2 r^{l+1} - (l+1) b_2 r^{-l} \right)
\]

\[
= \ell (l+1) a_2 r^l + \ell (l+1) b_2 r^{-l(l+1)} = \ell (l+1) R
\]

For \( \Theta \):

\[
\frac{1}{\sin \Theta} \frac{d}{d\Theta} (\sin \Theta \frac{d\Theta}{d\Theta}) - \frac{m^2}{\sin^2 \Theta} = -\ell (l+1)
\]

Let \( x = \cos \Theta \)

\[
dx = -\sin \Theta d\Theta \]

\[
d\Theta = -\frac{dx}{\sin \Theta}
\]

solutions for \(-1 \leq x \leq 1\) correspond to \( \ell \geq 0 \) integers

\[
\frac{d}{dx} \left[ (1-x^2) \frac{d\Theta}{dx} \right] + \left[ \ell (l+1) - \frac{m^2}{1-x^2} \right] \Theta = 0
\]

Called generalized Legendre Equation - solutions are called the associated Legendre functions.

Ordinary Legendre polynomials are solutions for \( m = 0 \).
For the special case \( m = 0 \), i.e. the solution has azimuthal symmetry and \( \Phi \) does not depend on the angle \( \Psi \) (i.e. rotational symmetry about \( z \) axis),

we want the solutions to

\[
\frac{1}{\rho} \left[ (1-\rho^2) \frac{d^2 \Phi}{d\rho^2} \right] + \ell(\ell+1) \Phi = 0
\]

The solutions are known as the Legendre polynomials, \( P_\ell(x) \).

They are given, for \( \ell \) integer, by

\[
P_\ell(x) = \frac{1}{2 \ell!} \left( \frac{d}{dx} \right)^\ell (x^2 - 1)^\ell
\]

Rodriguez's formula

The lowest \( \ell \) polynomials are

\[
P_0(x) = 1, \quad P_2(x) = \frac{1}{2} (3x^2 - 1),
\]

\[
P_1(x) = x, \quad P_3(x) = \frac{1}{2} (5x^3 - 3x)
\]

In general, \( P_\ell(x) \) is a polynomial of order \( \ell \) with only even powers of \( x \) if \( \ell \) is even, and only odd powers of \( x \) if \( \ell \) is odd. \( \Rightarrow \) \( P_\ell(x) \) is even in \( x \) for \( \ell \) even, odd in \( x \) for \( \ell \) odd

\( P_\ell(x) \) is normalized so that \( P_\ell(1) = 1 \)
Legendre polynomials are only for integer \( l \geq 0 \). What about solutions for non-integer \( l \)?

The \( P_l(x) \) give one solution for each integer \( l \).

But \( P_l(x) \) are defined by a second-order differential equation - shouldn't there be a second independent solution for each \( l \)?

It turns out that these "2nd" solutions, as well as solutions for non-integer \( l \), all blow up at either \( x = -1 \) or \( x = 1 \), i.e. at \( \theta = 0 \) or \( \theta = \pi \).

They therefore are physically unacceptable and we do not need to consider them. See Jackson 3.2.

The Legendre polynomials are orthogonal and form a complete set of basis functions on the interval \(-1 \leq x \leq 1\).

\[
\int_{-1}^{1} P_l(x) P_m(x) \, dx = \int_{0}^{\pi} \sin \theta P_l(\cos \theta) P_m(\cos \theta) \, d\theta = \begin{cases} 0 & l \neq m \\
\frac{2}{2l+1} & l = m \end{cases}
\]

\( \Rightarrow \) we can expand any function \( f(\theta) \), \( 0 \leq \theta \leq \pi \), as a linear combination of the \( P_l(\cos \theta) \).

This is the reason they are useful for solving problems of Laplace's type with spherical boundary surfaces.
For \( m \neq 0 \), the solutions to

\[
\frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} \phi \right] + \left[ \ell(\ell+1) - \frac{m^2}{1-x^2} \right] \phi = 0
\]

are the associated Legendre functions \( P_\ell^m(x) \).

For \( P_\ell^m(x) \) to be finite in interval \(-1 \leq x \leq 1\), one again finds that \( \ell \) must be integer \( \ell \geq 0 \), and integer \( m \) must satisfy \( |m| \leq \ell \), i.e. \( m = -\ell, -\ell+1, \ldots, 0, \ldots, \ell-1, \ell \). For each \( \ell \) and \( m \) there is only one such non-divergent solution.

It is typical to combine the solutions \( P_\ell^m(\cos \theta) \) to the \( \theta \)-part of the equation with the \( \Phi_\ell^m(\phi) = e^{im\phi} \) solutions to the \( \phi \)-part of the equation to define the spherical harmonics

\[
Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\phi}
\]

The \( Y_{\ell m} \) are orthogonal

\[
\int_0^{2\pi} \int_0^\pi \sin \theta \, d\theta \, d\phi \, Y_{\ell m}^* (\theta, \phi) \, Y_{\ell m} (\theta, \phi) = \delta_{\ell \ell'} \delta_{mm'}
\]

and are a complete set of basis functions for expanding any function \( f(\theta, \phi) \) defined on the surface of a sphere.
Behavior of fields near a circular hole or sharp tip.

We now want to solve the \( \nabla^2 \phi = 0 \) with separation of variables, but now \( \Theta \) is restricted to range \( 0 \leq \Theta \leq \beta \).

We still have azimuthal symmetry, but now, since we do not need solution to \( \phi \) to be finite for all \( \Theta \in [0, \pi] \), but only \( \Theta \in (0, \beta) \), we have more solutions to the \( \Theta \) equation, and it does not have to be integer. Still need \( L > 0 \) to be finite at \( \Theta = 0 \).

See Jackson sec. 3.4 for details.
Examples with azimuthal symmetry \( m = 0 \)

General solution to \( \nabla^2 \phi = 0 \) can be written in form

\[
\phi(r, \theta) = \sum_{\ell=0}^{\infty} \left[ A_{\ell} r^\ell + \frac{B_{\ell}}{r^{\ell+1}} \right] P_{\ell}(\cos \theta)
\]

determine the \( A_\ell \) and \( B_\ell \) from the boundary conditions of the particular problem.

1. Suppose one is given \( \phi(r, \theta) = \phi_0(\theta) \) on surface of sphere of radius \( R \).

To find solution of \( \nabla^2 \phi = 0 \) inside sphere

\( \phi \) should not diverge at origin \( \Rightarrow B_\ell = 0 \)

for all \( \ell \)

\[
\phi(r, \theta) = \sum_{\ell=0}^{\infty} A_{\ell} r^\ell P_{\ell}(\cos \theta)
\]

\[
\Rightarrow \phi(r, \theta) = \phi_0(\theta) = \sum_{\ell=0}^{\infty} A_{\ell} R^\ell P_{\ell}(\cos \theta)
\]

\[
\Rightarrow \int_0^\pi \sin \theta d\theta \phi_0(\theta) P_m(\cos \theta) = \sum_{\ell=0}^{\infty} A_{\ell} R^\ell \int_0^\pi \sin \theta P_{\ell}(\cos \theta) P_m(\cos \theta)
\]

\[
= \sum_{\ell=0}^{\infty} A_{\ell} R^\ell \left( \frac{2}{2\ell+1} \right) \delta_{\ell m}
\]

\[
A_m = \frac{2m+1}{2 R^m} \int_0^\pi \sin \theta d\theta \phi_0(\theta) P_m(\cos \theta)
\]

solution
To find solution of $V^2 \phi = 0$ outside sphere

1. require $\phi \to 0$ as $r \to 0$, then $A_0 = 0$ for all $l$

$$\phi(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

$$\phi(r, \theta) = \phi_r(\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

Gives solution

$$B_m = \frac{2m+1}{2} R^{m+1} \int_0^\pi \sin \theta \phi_r(\theta) P_m(\cos \theta) d\theta$$

$$B_m = A_m R^{2m+1}$$

2. Suppose one is given surface charge density $\sigma(\theta)$ fixed on surface of sphere of radius $R$.
What is $\phi$ inside and outside?

From previous example

$$\phi(r, \theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) & r < R \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta) & r > R \end{cases}$$

Boundary conditions at $r = R$ on surface

(i) $\phi$ continuous

$$\sum_{l=0}^{\infty} \left[ A_l r^l - \frac{B_l}{r^{l+1}} \right] P_l(\cos \theta) = 0$$
If an expansion in Legendre polynomials vanishes for all $\theta$, then each coefficient in the expansion must vanish

$$\Rightarrow \quad A_l \; R^{-l} = \frac{B_l}{R^{l+1}} \quad \Rightarrow \quad B_l = A_l \; R^{2l+1}$$

(ii) jump in electric field at $\sigma$

$$\frac{\partial \Phi^\text{out}}{\partial r} \bigg|_{r=R} + \frac{\partial \Phi^\text{in}}{\partial r} \bigg|_{r=R} = 4\pi \sigma$$

$$\Rightarrow \quad \sum_{l=0}^{\infty} \left[ \frac{(l+1) \; B_l}{R^{l+2}} + l \; A_l \; R^{l-1} \int P_l(\cos \theta) \right] = 4\pi \sigma$$

$$\Rightarrow \quad \sum_{l=0}^{\infty} \left[ \frac{(l+1) \; A_l \; R^{-2l+1}}{R^{l+2}} + l \; A_l \; R^{l-1} \int P_l(\cos \theta) \right] = 4\pi \sigma$$

$$\Rightarrow \quad \sum_{l=0}^{\infty} \left[ (2l+1) \; R^{-l-1} \; A_l \; P_l(\cos \theta) \right] = 4\pi \sigma$$

$$(2m+1) \; R^{-m-1} \; A_m \left( \frac{2}{2m+1} \right) = 4\pi \int_0^\pi d\theta \sin \theta \; \sigma(\theta) \; P_m(\cos \theta)$$

$$A_m = \frac{4\pi}{2 \; R^{m-1}} \int_0^\pi d\theta \sin \theta \; \sigma(\theta) \; P_m(\cos \theta)$$
Suppose \( \sigma(\Theta) = k \cos \Theta \) \quad \text{what is} \ \phi ?

Note \( \sigma(\Theta) = k \rho_1(\cos \Theta) \)

hence only \( A_1 \neq 0 \) by orthogonality of \( \rho_1(\cos \Theta) \)

\[
A_1 = \frac{4 \pi k}{2} \int_0^\frac{\pi}{2} \sin \Theta \rho_1(\cos \Theta) \rho_1(\cos \Theta) \ d\Theta
\]

\[
= \frac{4 \pi k}{2} \left( \frac{2}{2+1} \right) = \frac{4 \pi k}{3}
\]

\[\Rightarrow \phi(r, \Theta) = \begin{cases} 
\frac{4 \pi k r \cos \Theta}{3} & r < R \\
\frac{4 \pi k r^3}{3r^2} \cos \Theta & r > R
\end{cases}
\]

we will see that potential outside the sphere is that of an ideal dipole with dipole moment

\[p = \frac{4 \pi k r^3}{3}
\]

Inside the sphere, the potential \( \phi = \frac{4 \pi k \rho}{3} \)

where \( \rho = r \cos \Theta \). The electric field inside the sphere is therefore the constant

\[E = -\nabla \phi = -\frac{4 \pi k \rho}{3}
\]
outside the sphere the field is

\[
\mathbf{E} = -\nabla \phi = -\frac{\partial \phi}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta}
\]

\[
= \frac{8\pi}{3} k \frac{R^3}{r^3} \cos \theta \hat{r} + \frac{4\pi}{3} k \frac{R^3}{r^2} \sin \theta \hat{\theta}
\]

\[
\mathbf{E} = \frac{4\pi}{3} R^3 k \frac{1}{r^3} \left[ 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right]
\]

**dipole field**