Physical example with \( \sigma(\theta) = \kappa \cos \theta \)

Two spheres of radii \( R \), with equal but opposite uniform charge densities \( \rho \) and \(-\rho\), displaced by small distance \( d \ll R \)

Surface charge \( \sigma \) builds up due to displacement. This is a uniformly "polarized" sphere.

\[
\sigma(\theta) = \rho \, d \cos \theta
\]

Surface charge \( \sigma' \): \( \sigma(\theta) = \rho \, d \cos \theta \)

\[
d \cos \theta = Sr
\]

Total dipole moment is \( (pd) \frac{4}{3} \pi R^3 \)

Polarization = \( \frac{\text{dipole moment}}{\text{volume}} \) = \( \rho d \)

\( E \) field inside a uniformly polarized sphere is constant. \( E = -\rho d \frac{4\pi}{3} \)
(3) **Grounded Conducting Sphere in Uniform Electric Field** \( \mathbf{E} = \frac{E_0}{r} \hat{z} \)

As \( r \to \infty \) far from sphere, \( \mathbf{E} = E_0 \hat{z} \Rightarrow \phi = -E_0 z \)

Boundary Conditions:

\[
\begin{align*}
\phi(r, \theta) &= 0 \\
\phi(r \to \infty, \theta) &= -E_0 \cos \theta
\end{align*}
\]

The solution outside the sphere has the form:

\[
\phi(r, \theta) = \sum_{l=0}^{\infty} \left[ A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos \theta)
\]

From boundary condition as \( r \to \infty \) we have:

\[ A_l = 0 \quad \text{all } l \neq 1 \]

\[ A_1 = -E_0 \quad \text{since } P_1(\cos \theta) = \cos \theta \]

\[
\phi(r, \theta) = -E_0 \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)
\]

From \( \phi(r, \theta) = 0 \), we have:

\[
0 = -E_0 R \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta)
\]

\[ \Rightarrow B_l = 0 \quad \text{all } l \neq 1 \]

\[ B_1 = \frac{E_0 R}{R^2} \Rightarrow B_1 = \frac{E_0 R^3}{R^2} \]
\[ \phi(r, \theta) = -E_0 \left( r - \frac{R^3}{r^2} \right) \cos \theta \]

1st term is just potential \(-E_0 \cos \theta\) of the uniform applied electric field.

2nd term is potential due to the induced surface charge on the sphere – it is a dyole field.

Induced charge density is

\[
4\pi \sigma(\theta) = -\frac{\partial \phi}{\partial r} \bigg|_{r=R} = E_0 \left( 1 + 2\frac{R^3}{r^3} \right) \cos \theta = 3E_0 \cos \theta
\]

\[
\sigma(\theta) = \frac{3}{4\pi} E_0 \cos \theta \quad \text{like uniformly polarized sphere} \quad k = \frac{3E_0}{4\pi}
\]

From (2) we know that the field inside the sphere due to this \(\sigma\) is just

\[
-\frac{4}{3} \pi k \hat{z} = -\frac{4}{3} \pi \frac{3E_0}{4\pi} \hat{z}
\]

\[
= -E_0 \hat{z}. \text{ This is just what is required so that the total field in the conducting sphere vanishes.}
\]

Can check that outside the sphere, \(\vec{E} = -\nabla \phi\) is normal to surface of sphere at \(r = R\).
Multiply Expansion

region with \( p \neq 0 \)

We want to find the potential \( \phi \) for an arbitrary localized distribution of charge \( \rho \), at distances far away \( r \gg R \).

\[
\phi(r) = \int d^3r' \frac{\rho(r')}{|r-r'|}
\]

we want an expansion of \( \frac{1}{|r-r'|} \) in powers of \( \left( \frac{r'}{r} \right) \)

for \( r \gg r' \)

\[
\frac{1}{|r-r'|}
\]

view this as the potential at \( r \) due to a unit point charge located at position \( r' \).

We take \( r' \) on the \( z \) axis.

\[
\phi(r, \theta) = \sum_{\ell=0}^{\infty} \frac{B\ell}{r^{\ell+1}} P_{\ell}(\cos \theta)
\]

\[
\frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B\ell}{r^{\ell}} P_{\ell}(\cos \theta)
\]

The problem has azimuthal symmetry \( \Rightarrow \phi \) depends only on \( r \) and \( \theta \), so we can express it as an expansion in Legendre polynomials.

For \( r \gg r' \),

\[
\phi(r, \theta) = \sum_{\ell=0}^{\infty} \frac{B\ell}{r^{\ell+1}} P_{\ell}(\cos \theta)
\]

\[
\text{all } A_{\ell} = 0
\]

as need \( \phi \geq 0 \)

as \( r \to \infty \)
We know \( \phi(r, \theta = 0) = \frac{1}{r - r'} \) (for \( r > r' \))

\[ \Rightarrow \phi(r, 0) = \frac{1}{r} \sum_{l=0}^{\infty} \frac{B_l}{r^2} P_l(1) \]

\[ = \frac{1}{r} \sum_{l=0}^{\infty} \frac{B_l}{r^2} \quad \text{as } P_l(1) = 1 \]

\[ = \frac{1}{r} \left( \frac{1}{1 - \frac{r'}{r}} \right) \quad \text{exact result from Coulomb} \]

Now Taylor expansion \( \frac{1}{1 - \epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + \ldots \)

\[ \Rightarrow \frac{1}{r} \sum_{l=0}^{\infty} \frac{B_l}{r^2} = \frac{1}{r} \left( 1 + \left( \frac{r'}{r} \right) + \left( \frac{r'}{r} \right)^2 + \left( \frac{r'}{r} \right)^3 + \ldots \right) \]

\[ \Rightarrow B_l = \left( \frac{r'}{r} \right)^l \quad \text{is solution} \]

So for \( r > r' \)

\[ \frac{1}{|r - r'|} = \frac{1}{r} \sum_{l=0}^{\infty} \left( \frac{r'}{r} \right)^l P_l(\cos \theta) \]

So, for the charge contribution \( \rho \),

\[ \phi(r^2) = \int_0^{1} d^3r' \frac{\rho(r')}{|r - r'|} = \int_0^{1} d^3r' \frac{\rho(r')}{r^2} \sum_{l=0}^{\infty} \left( \frac{r'}{r} \right)^2 P_l(\cos \theta) \]

\[ = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int_0^{1} d^3r' \phi(r') \left( \frac{r'}{r} \right)^l P_l(\cos \theta) \]

where \( \theta \) is the angle between the fixed observation point \( r \) and the integration variable \( r' \).
This is the multipole expansion, which expresses the potential far from a localized source as a power series in \( r/r \). It is exact provided one adds all the infinite \( l \) terms. In practice, one generally approximates by summing only up to some finite \( l \).

Note: to do the integrals

\[
\int d^3r \ f(\vec{r}) \ (\vec{r})^2 \ \rho (\cos \theta)
\]

\( \theta \) is defined as the angle of \( \vec{r} \) with respect to observation point \( \vec{r}' \). We therefore in principle have to repeat the integration every time we change \( \vec{r} \).

We will find a way around this by

(i) just looking explicitly at the few lowest order terms

(ii) a general method involving spherical harmonics \( Y_l^m (\theta, \phi) \)
monopole: \( l=0 \) term

\[
\phi^{(0)}(\vec{r}) = \frac{1}{r} \int d^3r' \, f(r') \quad \text{where} \quad \phi = \int d^3r' \, f(r') \text{ is total charge}
\]

dipole: \( l=1 \) term

\[
\phi^{(1)}(\vec{r}) = \frac{1}{r^2} \int d^3r' \, \delta(\vec{r} - \vec{r}') \, \vec{r}' \cdot \vec{p} \, (\cos \theta)
\]

\[
\phi^{(1)}(\vec{r}) = \frac{1}{r^2} \int d^3r' \, f(\vec{r}') \, r' \, \cos \theta
\]

Now \( \vec{r} \cdot \vec{r}' = rr' \cos \theta \quad \Rightarrow \quad \vec{r} \cdot \vec{r}' = r' \cos \theta \)

\[
\phi^{(1)}(\vec{r}) = \frac{1}{r^2} \int d^3r' \, f(\vec{r}') \, \vec{r}'
\]

\[
\phi^{(1)}(\vec{r}) = \frac{\vec{p} \cdot \vec{r}}{r^2} \quad \text{where} \quad \vec{p} = \int d^3r' \, f(\vec{r}') \, \vec{r}'
\]

\( \vec{p} \) is the dipole moment

For a set of point charges \( q_i \) at \( \vec{r}_i \),

\[\vec{p} = \sum q_i \vec{r}_i\]
**Quadrupole: $l = 2$ Term**

\[
\phi^{(2)}(\vec{r}) = \frac{1}{r^2} \int d^3r' \rho(\vec{r}') r'^2 P_2(\cos \theta)
\]

\[
= \frac{1}{r^3} \int d^3r' \rho(\vec{r}') r'^2 \frac{1}{2} (3 \cos^2 \theta - 1)
\]

where \( \cos \theta = \hat{r} \cdot \vec{r} \).

\[
\phi^{(2)}(\vec{r}) = \frac{1}{r^3} \int d^3r' \rho(\vec{r}') \frac{1}{2} (3 (\vec{r}' \cdot \hat{r})^2 - (\vec{r}')^2)
\]

\[
= \frac{1}{r^3} \vec{r} \cdot \left[ \int d^3r' \rho(\vec{r}') \frac{1}{2} (3 \vec{r}' \cdot \hat{r}^2 - (\vec{r}')^2 \hat{l}) \right] \cdot \hat{r}
\]

where \( \hat{l} \) is the identity tensor such that for any two vectors \( \vec{u} \) and \( \vec{v} \), \( \vec{u} \cdot \hat{l} \cdot \vec{v} = \vec{u} \cdot \vec{v} \).

and \( \vec{r}' \vec{r} \) is the tensor such that for any two vectors \( \vec{u} \) and \( \vec{v} \), \( \vec{u} \cdot [\vec{r}' \vec{r}] \cdot \vec{v} = (\vec{u} \cdot \vec{r}') (\vec{r}' \cdot \vec{v}) \).

Define quadrupole tensor \( \mathcal{Q} \equiv \int d^3r' \rho(\vec{r}') (3\vec{r}' \vec{r} - (\vec{r}')^2 \hat{l}) \).

\[
\phi^{(2)}(\vec{r}) = \frac{1}{r^3} \frac{1}{2} \vec{r} \cdot \mathcal{Q} \cdot \hat{r}
\]

so to lowest three terms,

\[
\phi(\vec{r}) = \frac{\rho}{r} + \frac{\vec{p} \cdot \hat{r}}{r^2} + \frac{\hat{r} \cdot \mathcal{Q} \cdot \hat{r}}{2r^3} + \ldots
\]

defined in terms of the moments \( \rho, \vec{p}, \mathcal{Q} \) of the charge distribution.
Note, the moments \( \mathbf{m} \), \( \mathbf{P} \), \( \mathbf{E} \) do not depend on the observation point \( \mathbf{r} \) — we can calculate them once and then use them to get \( \Phi(\mathbf{r}) \) at all \( \mathbf{r} \).

**Monopole**: \( \Phi = \int d^3r \, p(\mathbf{r}) \) — scalar integral

**Dipole**: \( \mathbf{P} = \int d^3r \, p(\mathbf{r}) \mathbf{r} \) — vector integral

If we pick a coordinate system, we have to do 3 integrations to get the three components of \( \mathbf{P} \)

\( \hat{e}_i \cdot \mathbf{P} = p_i = \int d^3r \, p(\mathbf{r}) r_i \)

**Quadrapole**: \( \mathbf{Q} = \int d^3r \, p(\mathbf{r}) \left( \mathbf{3r} \mathbf{r} - (\mathbf{r} \cdot \mathbf{r}) \mathbf{I} \right) \) — tensor integral

If we pick a coordinate system \( x, y, z \) then

\( \mathbf{Q} \) is a matrix with components

\( \hat{e}_i \cdot \mathbf{Q} \hat{e}_j = Q_{ij} = \int d^3r \, p(\mathbf{r}) \left[ 3r_i r_j - (\mathbf{r} \cdot \mathbf{r}) \delta_{ij} \right] \)

There are 9 elements of the 3x3 matrix \( (Q_{ij}) \), but \( Q_{ij} = Q_{ji} \) is symmetric so there are only 6 independent elements to compute.
General method

\[ \phi (r') = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int d^3r' \rho (r')(r')^l P_l (\cos \theta) \]

In above, \( \theta \) is angle between \( \hat{r} \) and \( \hat{r}' \).
If we think of \( \rho \) as the spherical coord \( \theta \), then we effect, above is choosing \( \hat{r} \) to be on \( \hat{z} \) axis. We would like a representation in which \( \hat{r} \) is positioned arbitrarily with respect to the axes used in describing \( \rho \).

Use the addition theorem for spherical harmonics
- see Jackson 3.6 for discussion & proof

\[ P_l (\cos \theta) = \frac{4 \pi}{2l+1} \sum_{m=-l}^{l} Y_{l}^{m} (\theta, \phi) Y_{l}^{m} (\theta, \phi') \]

where \((\theta, \phi)\) are the angles of \( \hat{r} \), \((\theta', \phi')\) are the angles of \( \hat{r}' \), and \( \theta \) is the angle between \( \hat{r} \) and \( \hat{r}' \), i.e. \( \cos \theta = \hat{r} \cdot \hat{r}' \)
\[ \cos \theta = \frac{\hat{z} \cdot \hat{r}}{\hat{z} \cdot \hat{r}'} \]
\[ \cos \theta' = \frac{\hat{z} \cdot \hat{r}'}{\hat{z} \cdot \hat{r}} \]

\[ \Rightarrow \]

\[ \phi (r') = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \frac{4 \pi}{2l+1} \sum_{m=-l}^{l} \int d^3r' \rho (r')(r')^l \frac{l}{2 \pi} Y_{l}^{m} (\theta', \phi') Y_{l}^{m*} (\theta, \phi) \]

Define the moment

\[ \mathbf{r}_{l m} = \int d^3r' \rho (r')(r')^l Y_{l}^{m} (\theta', \phi') \]

independent of observation point
\[ \phi(r) = \frac{q}{r} + \frac{p}{r^2} + \frac{\mathbf{e} \cdot \mathbf{e}_0}{2r^3} \]

Electric field \( \mathbf{E} = -\nabla \phi = -\frac{\partial \phi}{\partial r} \mathbf{r} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \mathbf{\hat{\theta}} = -\frac{\partial \phi}{r \sin \theta} \mathbf{\hat{\phi}} \]

For the monopole term \( \mathbf{E} = \frac{q}{r^2} \mathbf{r} \)

For the dipole term, choose \( \mathbf{p} \) along \( \hat{\phi} \) axis so

\[ \phi(r) = \frac{p \cos \theta}{r^2} \]

\[ \mathbf{E} = \frac{2p \cos \theta}{r^3} \mathbf{\hat{r}} + \frac{p \sin \theta}{r^3} \mathbf{\hat{\theta}} \]

\[ \mathbf{E} = \frac{p}{r^3} (2 \cos \theta \mathbf{\hat{r}} + \sin \theta \mathbf{\hat{\theta}}) \]

Note \( p \cos \theta \mathbf{\hat{r}} = (\mathbf{p} \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}} \)

\( p \sin \theta \mathbf{\hat{\theta}} = - (\mathbf{p} \cdot \mathbf{\hat{\theta}}) \mathbf{\hat{\theta}} \)

Now \( \mathbf{p} = (\mathbf{p} \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}} + (\mathbf{p} \cdot \mathbf{\hat{\theta}}) \mathbf{\hat{\theta}} \)

\[ - (\mathbf{p} \cdot \mathbf{\hat{\theta}}) \mathbf{\hat{\theta}} = (\mathbf{p} \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}} - \mathbf{p} \]

\[ \mathbf{E} = \frac{1}{r^3} \left[ 2(\mathbf{p} \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}} + (\mathbf{p} \cdot \mathbf{\hat{\theta}}) \mathbf{\hat{\theta}} - \mathbf{p} \right] \]

\( = \frac{1}{r^3} \left[ 2(\mathbf{p} \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}} - \mathbf{p} \right] \) expresses \( \mathbf{E} \) in coordinate-free form