Example: circular current loop in xy plane

\[ \text{radius } R \]

\[ \vec{B} \]

for \( r > R \), \( \vec{\nabla} \times \vec{B} = 0 \) \( \implies \vec{B} = -\vec{\nabla} \phi_M \)

where \( \nabla^2 \phi_M = 0 \).

Try Legendre polynomial expansion for \( \phi_M \)

\[ \phi_M = \sum_{\ell=0}^{\infty} \frac{B_{\ell}}{r^{\ell+1}} P_\ell (\cos \theta) \]

(\( A_\ell \) terms vanish as we want \( R \to 0 \) as \( r \to \infty \))

\[ \vec{B} = -\vec{\nabla} \phi_M = -\frac{\partial \phi_M}{\partial \vec{r}} \hat{\vec{r}} - \frac{1}{r} \frac{\partial \phi_M}{\partial \theta} \hat{\vec{\theta}} \]

\[ = \sum_{\ell} \frac{(\ell+1)B_{\ell}}{r^{\ell+2}} P_\ell (\cos \theta) \hat{\vec{r}} - \frac{B_{\ell}}{r^{\ell+2}} \frac{2 \ell \cos \theta}{\partial \theta} \hat{\vec{\theta}} \]

write \( \frac{\partial P_\ell}{\partial \theta} = \frac{\partial P_\ell}{\partial x} \frac{\partial x}{\partial \theta} = -\frac{\partial P_\ell}{\partial x} \sin \theta = -P_\ell' \sin \theta \)

\[ \vec{B} = \sum_{\ell} \frac{(\ell+1)B_{\ell}}{r^{\ell+2}} P_\ell (\cos \theta) \hat{\vec{r}} + \frac{B_{\ell}}{r^{\ell+2}} \sin \theta P_\ell' (\cos \theta) \hat{\vec{\theta}} \]

To determine the \( \vec{d} \vec{B} \) we compare with exact solution along \( \hat{\vec{\theta}} \) axis

\[ \vec{B} (\hat{\vec{\theta}}, \hat{\vec{r}}) = \sum_{\ell} \frac{(\ell+1)B_{\ell}}{r^{\ell+2}} \hat{\vec{r}} = \sum_{\ell} \frac{(\ell+1)B_{\ell}}{r^{\ell+2}} \hat{\vec{\theta}} \]

Since \( P_\ell (1) = 1 \), \( \sin (0) = 0 \) and \( P_\ell' (1) \) finite, \( \hat{\vec{r}} = \hat{\vec{\theta}} \)

with \( \beta = \theta \).
exact solution on \( \frac{\partial}{\partial z} \) axis:

\[
\vec{A} = \int \frac{d^3r'}{\varepsilon} \vec{f}(r') \Rightarrow \vec{B}(r) = \nabla \times \vec{A} = \int \frac{d^3r'}{\varepsilon} \nabla \times \frac{\vec{f}(r')}{|r-r'|^3}
\]

\[
\vec{B} = -\int \frac{d^3r'}{\varepsilon} \vec{f}(r') \times \vec{v} \left( \frac{1}{|r-r'|^3} \right)
\]

\[
\vec{B} = \int \frac{d^3r'}{\varepsilon} \vec{f}(r') \times (\vec{r}-\vec{r'}) \left( \frac{1}{|r-r'|^3} \right) \quad \text{Biot-Savart law for magnetostatics}
\]

For our loop

\[
\vec{B}(z) = \int_0^{2\pi} \frac{d\phi}{\varepsilon} \frac{\vec{r} \times \vec{\phi}}{c} \left( \frac{-R \vec{\hat{r}} + z \vec{\hat{z}}}{(z^2+R^2)^{3/2}} \right)
\]

\[
\vec{r} \times \vec{\phi} = \hat{z}
\]

\[
\frac{2\pi}{\varepsilon} \frac{R (IR)}{c (z^2+R^2)^{3/2}}
\]

\[
\vec{B}(z) = \frac{2\pi R^2 I}{c} \frac{z}{(z^2+R^2)^{3/2}}
\]


to match Legendre polynomial expansion, do Taylor series expansion above

\[
\vec{B}(z) = \frac{2\pi R^2 I}{c} \frac{z}{3} \left( \frac{1}{(1+(\frac{R}{z})^2)^{3/2}} \right) = \frac{2\pi R^2 I}{c} \frac{z}{3} \left\{ 1 - \frac{3}{2} \left( \frac{R}{z} \right)^2 + \cdots \right\}
\]

\[
= \frac{2\pi R^2 I}{c} \frac{z}{3} \left\{ \frac{1}{3} - \frac{3}{2} \frac{R^2}{z^3} + \cdots \right\}
\]

\[
= \left\{ \frac{B_0}{3^2} + \frac{2 B_1}{3^3} + \frac{3 B_2}{3^4} + \frac{4 B_3}{3^5} + \cdots \right\} \frac{z}{3}
\]
\[ B_0 = 0, \quad B_1 = \frac{\pi R^2 I}{c}, \quad B_2 = 0, \quad B_3 = -\frac{3\pi R^2 I R^2}{4c} \]

So to order \( L = 3 \)

\[
\vec{B}(r) = \frac{\pi R^2 I}{c} \left\{ \frac{2 P_1(\cos \theta) \hat{r} + \sin \theta P_1'(\cos \theta) \hat{\theta}}{r^3} \right. \\
\left. - \frac{\left[ 3R^2 P_3(\cos \theta) \hat{r} + \frac{3}{4} R^2 \sin \theta P_3'(\cos \theta) \hat{\theta} \right]}{r^5} \right\} + \ldots \]

\[ P_1(x) = x \quad \Rightarrow \quad P_1'(x) = 1 \]
\[ P_3(x) = \frac{1}{2}(5x^3 - 3x) \quad \Rightarrow \quad P_3'(x) = \frac{1}{2}(15x^2 - 3) \]

\[
\vec{B}(r) = \frac{\pi R^2 I}{c} \left\{ \frac{2 \cos \theta \hat{r} + \sin \theta \hat{\theta}}{r^3} \right. \\
\left. - \left[ \frac{3}{2} R^2 (5 \cos^3 \theta - 3 \cos \theta) \hat{r} + \frac{3}{8} R^2 \sin \theta (15 \cos^2 \theta - 3) \hat{\theta} \right] \right\} + \ldots \]

\[ \frac{\pi R^2 I}{c} = m \] is the magnetic dipole moment of the loop.

We see that the 1st term is just the magnetic dipole moment term. The 2nd term is the magnetic quadrupole term. Could easily get higher order terms by this method.

Compare our result above to Jackson (5.40)
Symmetry under parity transformation

**vector vs. pseudo vector**

\[ \vec{r} = (x, y, z) \quad \rightarrow \quad (-x, y, -z) \]

\[ \mathcal{P}(\vec{r}) = -\vec{r} \quad \text{position } \vec{r} \text{ is odd under parity} \]

Any vector-like quantity that is odd under \( \mathcal{P} \) is a **vector**.

**Angles of Vectors**

- Position \( \vec{r} \)
- Velocity \( \vec{v} = \frac{d\vec{r}}{dt} \) since \( \vec{v} \) is vector and \( t \) is scalar \( \mathcal{P}(t) = t \)
- Acceleration \( \vec{a} = \frac{d\vec{v}}{dt} \)
- Force \( \vec{F} = m\vec{a} \) since \( \vec{a} \) is vector and \( m \) is scalar
- Momentum \( \vec{p} = m\vec{v} \) since \( \vec{v} \) is vector and \( m \) is scalar
- Electric field \( \vec{E} = \mathcal{E} \vec{E} \) since \( \vec{E} \) is vector and \( \mathcal{E} \) is scalar \( \mathcal{P}(\mathcal{E}) = \mathcal{E} \)
- Current \( \vec{j} = \sum_i f_i \vec{v}_i \) since \( \vec{v}_i \) is vector and \( f_i \) is scalar
any vector-like quantity that is even under $P$ is a **pseudo-vector**

angular momentum $\vec{L} = \vec{r} \times \vec{p}$ since $\vec{r} \rightarrow -\vec{r}$ and $\vec{p} \rightarrow \vec{p}$, $\vec{L} \rightarrow -\vec{L}$ under $P$

magnetic field $\vec{F} = q \vec{v} \times \vec{B}$ since $\vec{F}$ and $\vec{v}$ are vectors and $q$ is scalar, $\vec{B}$ must be pseudo-pseudo-vector,

cross product of any two vectors is a pseudo-vector

"vector" and pseudo-vector is a vector

When solving for $\vec{E}$, it can only be made up of [vectors] that exist in the problem
When solving for $\vec{B}$, it can only be made up of [pseudo-vectors] that exist in the problem

\[ \nabla \cdot \vec{E} = \text{charged plane} \]

\[ \nabla \times \vec{E} = \hat{n} \]

only directions in problem is normal $\hat{n}$

$\hat{n}$ is a vector

\[ \vec{E} \propto \hat{n} \]

\[ \text{surface current} \]

\[ \frac{\partial \vec{E}}{\partial t} \]

only directions are the vectors $\hat{m}$ and $\hat{n}$. But $\vec{B}$ can only be made of pseudo-vectors

\[ \Rightarrow \vec{B} \propto (\hat{k} \times \hat{m}) \]
Dielectrics + Magnetic Materials - Macroscopic Maxwell Equations

Maxwell's equations apply exactly to the true microscopic electric and magnetic fields that arise from all charges and currents.

\[ \nabla \cdot \mathbf{E} = \mathbf{0} \quad \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \]

\[ \nabla \cdot \mathbf{B} = 4\pi \rho \quad \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + c \frac{\partial \mathbf{E}}{\partial t} \]

where \( \mathbf{E} \) and \( \mathbf{B} \) are microscopic fields from total charge density \( \rho \) and current density \( \mathbf{j} \).

However, in most problems involving macroscopic objects, if we took \( \rho \) and \( \mathbf{j} \) to describe charge and current of each individual atom in a material, then they, and the resulting \( \mathbf{E} \) and \( \mathbf{B} \) would be enormously complicated functions varying rapidly over distances \( \sim 10^{-8} \) cm and times \( \sim 10^{-16} \) sec.

In classical E&M we are generally concerned with phenomena that vary extremely slowly compared to these length and time scales,
Rather than worry about the microscopic details of $\mathbf{\nabla}$ and $\mathbf{j}$ as resulting $\mathbf{E}$ and $\mathbf{B}$ we want to describe phenomena in terms of averaged smoothly varying, averaged quantities that are smoothly varying at the atomic scale. The results in what are known as the macroscopic Maxwell equations.

**Dielectric Materials**

Can be solid, liquid, or gas.

A dielectric material is an insulator. Electrons are bound to the ionic cores of the atoms. When no electric field is present, the averaged $\mathbf{p}$ in the dielectric vanishes! One might therefore think that electrodynamics in a dielectric is just due to whatever "extra" or "free" charge is added to the dielectric. However, this is not true due to the phenomena of "polarization."

\[
\begin{align*}
E = 0 & \quad \mathbf{E > 0} \\
\begin{array}{c}
\text{electron cloud} \\
\text{centered on ionic core} \\
\text{depolle moment vanishes}
\end{array} & \quad \begin{array}{c}
\text{electron cloud} \\
\text{and ionic core rotated} \\
\text{atomic polarizability}
\end{array} \\
\end{align*}
\]

\[
\mathbf{p} = \alpha \mathbf{E} \\
\mathbf{j} = q \mathbf{E}
\]