Polarization

Consider a transverse plane wave traveling in direction \( \hat{n} \), i.e. \( \vec{k} = k \hat{n} \). Define a right-handed coordinate system as follows:

\[
\hat{e}_1 \times \hat{e}_2 = \hat{n} \\
\hat{m} \times \hat{e}_1 = \hat{e}_2 \\
\hat{e}_2 \times \hat{m} = \hat{e}_1
\]

A general solution to Maxwell's equations for a transverse plane wave is then

\[
\vec{E}(\vec{r}, t) = \Re \{ \vec{E}_1 \hat{e}_1 + \vec{E}_2 \hat{e}_2 \} e^{i(k \cdot \vec{r} - \omega t)}
\]

\[
\vec{H}(\vec{r}, t) = \frac{c}{w \mu} \Re \{ k \hat{m} \times (\vec{E}_1 \hat{e}_1 + \vec{E}_2 \hat{e}_2) \} e^{i(k \cdot \vec{r} - \omega t)}
\]

\[
= \frac{c}{w \mu} \Re \{ k (\vec{E}_1 \hat{e}_2 - \vec{E}_2 \hat{e}_1) \} e^{i(k \cdot \vec{r} - \omega t)}
\]

In general, \( k \) is complex

\[
k = k_1 + i k_2 = |k| e^{i \delta}
\]

So far we implicitly assumed that \( E_1 \) and \( E_2 \) are real constants. In this case

\[
\vec{E}(\vec{r}, t) = \vec{E}_0 e^{-k_2 \hat{m} \cdot \vec{r}} \cos (k_1 \hat{m} \cdot \vec{r} - \omega t)
\]

\[
\vec{H}(\vec{r}, t) = \vec{H}_0 e^{-k_2 \hat{m} \cdot \vec{r}} \cos (k_1 \hat{m} \cdot \vec{r} - \omega t + \delta)
\]

where

\[
\vec{E}_0 = \vec{E}_1 \hat{e}_1 + \vec{E}_2 \hat{e}_2 \quad \text{and} \quad \vec{H}_0 = \frac{c |k|}{w \mu} (\vec{E}_1 \hat{e}_2 - \vec{E}_2 \hat{e}_1)
\]

are fixed vectors for all time and space.
In this case the directions of $\hat{E}$ and $\hat{H}$ remain fixed while the amplitudes oscillate in time and space. Such a plane wave is called a linearly polarized wave.

However, there is nothing to prevent one from choosing a solution with $E_1$ and $E_2$ complex numbers:

$$E_1 = |E_1| e^{iX_1}, \quad E_2 = |E_2| e^{iX_2}$$

In this case one has

$$\vec{E}(r,t) = \text{Re} \left\{ |E_1| \hat{e}_1 e^{i(k \cdot \hat{r} - \omega t + X_1)} + |E_2| \hat{e}_2 e^{i(k \cdot \hat{r} - \omega t + X_2)} \right\}$$

and

$$\vec{H}(r,t) = \frac{c|k|}{\omega \mu} \text{Re} \left\{ |E_1| \hat{e}_2 e^{i(k \cdot \hat{r} - \omega t + S + X_1)} - |E_2| \hat{e}_1 e^{i(k \cdot \hat{r} - \omega t + S + X_2)} \right\}$$

unless $X_1 = X_2$ we see that the components of $\hat{E}$ and $\hat{H}$ in directions $\hat{e}_1$ and $\hat{e}_2$ will oscillate out of phase with each other. Thus the directions of $\hat{E}$ and $\hat{H}$ will oscillate in time and space, as well as the amplitudes of $\hat{E}$ and $\hat{H}$. The direction of $\hat{E}$ and $\hat{H}$ is no longer fixed.
we will see that this situation in general corresponds to \textit{elliptically polarized} wave!

\textbf{General case} \quad E_1, \text{ and } E_2 \text{ are complex constants. Write } \quad E_1 \hat{e}_1 + E_2 \hat{e}_2 = \tilde{U} e^{i\phi} \quad \text{where } \phi \text{ is chosen so that } \tilde{U} \cdot \tilde{U} \text{ is real}

\text{one can always do this since } \quad \tilde{U} \cdot \tilde{U} = (E_1^2 + E_2^2) e^{2i\phi} \quad \text{so } 2\phi \text{ is just the phase of the complex } E_1^2 + E_2^2

\tilde{U} \text{ is a complex vector } \Rightarrow \tilde{U} = \tilde{U}_a + i \tilde{U}_b \quad \text{with } \tilde{U}_a \text{ and } \tilde{U}_b \text{ real vectors.}

\text{Since } \tilde{U} \cdot \tilde{U} \text{ is real } \Rightarrow \tilde{U}_a \cdot \tilde{U}_b = 0

\text{so } \tilde{U}_a \perp \tilde{U}_b \text{ orthogonal}

\text{let } \hat{e}_a \text{ be the unit vector in direction of } \tilde{U}_a \quad \text{so } \tilde{U}_a = U_a \hat{e}_a \quad \text{with } U_a = |\tilde{U}_a| \quad \text{let } \hat{e}_b = \hat{\mathbf{m}} \times \hat{e}_a \quad \text{so that } \{\hat{\mathbf{m}}, \hat{e}_a, \hat{e}_b\} \text{ are a right handed coordinate system}

Then \quad \tilde{U}_b = \pm U_b \hat{e}_b \quad \text{where} \quad U_b = |\tilde{U}_b| \quad \text{since } \tilde{U}_b \perp \tilde{U}_a \text{ and both } \text{are } \perp \text{ to } \hat{\mathbf{m}}.

It is \((+)\) if \(\tilde{U}_b \) is parallel to \(\hat{e}_b\) and it is \((-)\) if \(\tilde{U}_b \) is antiparallel to \(\hat{e}_b\).
In this representation we have

\[ \hat{E}(\hat{r},t) = \text{Re} \left\{ \hat{u} e^{i \phi} e^{i \left( \hat{k} \cdot \hat{r} - \omega t \right)} \right\} \]

\[ = e^{-k_z \hat{z} \cdot \hat{r}} \text{Re} \left\{ \hat{u}_a \hat{e}_a e^{i \left( k_z \hat{n} \cdot \hat{r} - \omega t + \phi \right)} \right. \]

\[ + \hat{u}_b \hat{e}_b (\hat{e} \cdot \hat{r}) e^{i \left( k_z \hat{n} \cdot \hat{r} - \omega t + \phi \right)} \right\} \]

\[ = e^{-k_z \hat{z} \cdot \hat{r}} \left\{ \hat{u}_a \hat{e}_a \cos (\varphi + \phi) + \hat{u}_b \hat{e}_b \sin (\varphi + \phi) \right\} \]

where we write \( \varphi \equiv k_z \hat{n} \cdot \hat{r} - \omega t \)

Let's define \( e^{-k_z \hat{z} \cdot \hat{r}} \hat{u}_a \rightarrow \hat{u}_a \)

\( e^{-k_z \hat{z} \cdot \hat{r}} \hat{u}_b \rightarrow \hat{u}_b \)

so we don't have to keep writing the constant attenuation factor that is a common factor of all components of \( \hat{E} \).

Then define \( E_a \) and \( E_b \) as the components of \( \hat{E} \) in the directions \( \hat{e}_a \) and \( \hat{e}_b \) respectively.

\[ E_a = \hat{u}_a \cos (\varphi + \phi) \]

\[ E_b = \pm \hat{u}_b \sin (\varphi + \phi) \]

This then gives

\[ \left( \frac{E_a}{u_a} \right)^2 + \left( \frac{E_b}{u_b} \right)^2 = \cos^2 (\varphi + \phi) + \sin^2 (\varphi + \phi) = 1 \]

This is just the equation for an ellipse.
with semi-axes of lengths \( U_a \) and \( U_b \), oriented in the directions of \( \hat{e}_a \) and \( \hat{e}_b \).

![Diagram showing an ellipse with vectors](image)

\( \Rightarrow \) At a fixed position \( \vec{r} \), the tip of the vector \( \vec{E} \) will trace out the above ellipse as the time increases by one period of oscillation \( 2\pi/w \).

For (\( + \)), i.e. \( \vec{U}_b = U_b \hat{e}_b \), \( \vec{E} \) goes around the ellipse counterclockwise as \( t \) increases.

For (\( - \)), i.e. \( \vec{U}_b = -U_b \hat{e}_b \), \( \vec{E} \) goes around the ellipse clockwise as \( t \) increases.

Such a wave is said to be elliptically polarized.

Special cases:

1. \( U_a = 0 \) or \( U_b = 0 \)
   - The wave is linearly polarized.
(2) $U_a = U_b$

The tip of $\mathbf{E}$ traces out a circle as $t$ increases. The wave is circularly polarized.

The (+) case is said to have right-handed circular polarization.

The (−) case is said to have left-handed circular polarization.

One can define circular polarization basis vectors

$$\mathbf{\hat{e}_+} = \mathbf{\hat{e}_a} + i \mathbf{\hat{e}_b} \quad \frac{1}{\sqrt{2}}$$
$$\mathbf{\hat{e}_-} = \mathbf{\hat{e}_a} - i \mathbf{\hat{e}_b} \quad \frac{1}{\sqrt{2}}$$

with $\mathbf{\hat{e}_a}$ and $\mathbf{\hat{e}_b}$ orthonormal.

A wave with amplitude $\mathbf{E}_w = E \mathbf{\hat{e}_+}$ is right-handed circularly polarized.

A wave with complex amplitude $\mathbf{E}_w = E \mathbf{\hat{e}_-}$ is left-handed circularly polarized.

Just as the general case can always be written as a superposition of two orthogonal linearly polarized waves, i.e.

$$\mathbf{E}_w = E_1 \mathbf{\hat{e}_1} + E_2 \mathbf{\hat{e}_2}$$
one can also always write the general case as a superposition of a left handed and a right handed circularly polarized wave

\[ \mathbf{\hat{U}} = \mathbf{U}_a + i \mathbf{U}_b = \mathbf{U}_a \mathbf{\hat{e}}_a \pm i \mathbf{U}_b \mathbf{\hat{e}}_b \]

\[ = \left( \frac{\mathbf{U}_a + \mathbf{U}_b}{\sqrt{2}} \right) \mathbf{\hat{e}}_\pm + \left( \frac{\mathbf{U}_a - \mathbf{U}_b}{\sqrt{2}} \right) \mathbf{\hat{e}}_\mp \]

(Expand substitute in for \( \mathbf{\hat{e}}_\pm \) and expand, to see that this is so)

\[ \Rightarrow \text{An elliptically polarized wave can be written as a superposition of circularly polarized waves} \]

As a special case of the above (if \( \mathbf{U}_a = 0 \) or \( \mathbf{U}_b = 0 \)) a linearly polarized wave can always be written as a superposition of circularly polarized waves.
Reflection & Transmission of waves at Interfaces

Consider wave propagating from medium A into medium B.

For simplicity assume $E_0$ is real and positive, $E_0$ may be complex.

$\mu_a$ and $\mu_b$ are real and constant.

The incident wave, $\theta_i$ = angle of incidence
The reflected wave, $\theta_i$ = angle of reflection
The transmitted or "refracted" wave, $\theta_i$ = angle of refraction

Let each wave be given by

$$\hat{F}_n(\hat{r}, t) = \hat{F}_n e^{-i(k_n \hat{r} - \omega_n t)}$$

Where $\hat{F}_n$ can be either $E_n$ or $H_n$ for the electric or magnetic component of the wave.

Boundary condition: tangential component $E$

must be continuous at $z = 0$. If $\vec{E}$ is a vector in xy plane, and we consider $\vec{F} = 0$, then

$$\vec{E}_2 = \vec{E}_1 e^{-i\omega t}$$

must be true for all time. Can only happen if

$$\omega_0 = \omega_1 = \omega_2 = \omega$$

all frequencies are equal.
Now consider the same boundary condition for \( \vec{n} \) a position vector in the \( xy \) plane at \( z = 0 \). Since \( \omega / c \) is all equal, we can cancel out the common \( e^{i\omega t} \) factors to get

\[
\hat{\vec{n}} \cdot \vec{E}_0 e^{-i \vec{k}_0 \cdot \vec{n}} + \hat{\vec{n}} \cdot \vec{E}_1 e^{-i \vec{k}_1 \cdot \vec{n}} = \hat{\vec{n}} \cdot \vec{E}_2 e^{-i \vec{k}_2 \cdot \vec{n}}
\]

this must be true for all \( \vec{n} \). Can only happen if the projections of the \( \vec{k}_n \) in the \( xy \) plane are all equal.

\[
\begin{align*}
  k_{0x} &= k_{1x} = k_{2x} \\
  k_{0y} &= k_{1y} = k_{2y}
\end{align*}
\]

only 3 components of vectors can be different

Choose coordinate system as in diagram so that all \( \vec{k} \) vectors lie in the \( xy \) plane (\( y \) is out of page)

Since \( \varepsilon_0 \) is real and positive, \( \vec{E}_0 \) and \( \vec{E}_1 \) are real vectors

\[
k_{0x} = k_{1x} \Rightarrow |\vec{k}_0| \sin \theta_0 = |\vec{k}_1| \sin \theta_1
\]

Since \( k_0^2 = \frac{\omega^2}{c^2} \mu_0 \varepsilon_0 \) and \( k_1^2 = \frac{\omega^2}{c^2} \mu_0 \varepsilon_0 \),

then \( |\vec{k}_0| = |\vec{k}_1| \) so \( \sin \theta_0 = \sin \theta_1 \)

\[
\theta_0 = \theta_1
\]

angle of incidence = angle of reflection
If $\beta_b$ is also real and positive (B is transparent), then $|k_2|$ is real

$$k_{0x} = k_{2x} \Rightarrow |k_0| \sin \theta_0 = |k_2| \sin \theta_2$$

$$k_2^2 = \frac{\omega^2}{c^2} - \mu_b \varepsilon_b$$

$$\Rightarrow \sqrt{\mu_a \varepsilon_a} \sin \theta_0 = \sqrt{\mu_b \varepsilon_b} \sin \theta_2$$

in terms of index of refraction $M = k_c = \frac{\omega \sqrt{\mu \varepsilon}}{c}$

$$M = \sqrt{\mu \varepsilon}$$

$$\Rightarrow \frac{\sin \theta_2}{\sin \theta_0} = \frac{M_a}{M_b} \quad \text{Snell's Law}$$

true for all types of waves, not just EM waves

If $M_a > M_b$ then $\theta_2 > \theta_0$.

In this case, when $\theta_0$ is too large, we will have $\frac{M_a \sin \theta_0}{M_b} > 1$ and there will be no solution for $\theta_2$.

$\Rightarrow$ no transmitted wave

This is "total internal reflection" - wave does not exit medium A. The critical angle, above which one has total internal reflection, is given by

$$\frac{M_a}{M_b} \sin \theta_c = 1 \Rightarrow \theta_c = \arcsin \left( \frac{M_b}{M_a} \right)$$