Electromagnetism

Clearly \( E + B \) must transform into each other under Lorentz
transform.

in material frame \( K \)
stationary line charge \( q \)

\[
\begin{align*}
E & \approx \frac{q}{r^2} \\
& \text{cylindrical outward electric field}
\end{align*}
\]

in frame \( K' \) moving with \( \vec{v} \parallel \vec{E} \) to wire

\[
\begin{align*}
\text{moving line charge gives current} \\
\Rightarrow B \text{ circulating around wire} \\
as well as outward radial} \vec{E}
\end{align*}
\]

Lorentz force
\[
\vec{F} = q \vec{E} + q \frac{\vec{v} \times \vec{B}}{c}
\]

What is the velocity \( \vec{v} \) here? velocity with respect to
what material frame? Clearly \( \vec{E} \) and \( \vec{B} \) must change
from material frame to another if this force law

can make sense.

Charge density

Consider charge \( \Delta q \) contained in a vol \( \Delta V \).
\( \Delta q \) is a Lorentz invariant scalar.

Consider the reference frame in which the charge
is instantaneously at rest. In this frame
\[ \Delta Q = \hat{\rho} \Delta V \]

\( \hat{\rho} \) is charge density in the rest frame
\( \Delta V \) is volume in the rest frame

\( \hat{\rho} \) is Lorentz invariant by definition

Now transform to another frame moving with \( \vec{v} \) with respect to rest frame

\( \Delta Q \) remains the same
\( \Delta V = \frac{\Delta V'}{\gamma} \) volume contracts in direction \( \perp \) to \( \vec{v} \)

\[ \hat{\rho} = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q}{\Delta V} \gamma = \hat{\rho} \gamma \]

Current density is \( \vec{J} = \hat{\rho} \vec{v} = \gamma \vec{v} \cdot \vec{p} = \hat{\rho} \vec{u} \)

Define 4-current

\[ \vec{j}_\mu = (\vec{J}, \epsilon \gamma \vec{p}) = \hat{\rho} (\vec{u}, \epsilon \gamma) \]

\[ \frac{\partial \vec{j}_\mu}{\partial x^\mu} = 0 \]

it is 4-vector since \( \vec{u}_\mu \) is 4-vector and \( \hat{\rho} \) is Lorentz invariant scalar.

Charge conservation

\[ \nabla \cdot \vec{J} + \frac{\partial \hat{\rho}}{\partial t} = \frac{\partial \vec{j}_\mu}{\partial x^\mu} = 0 \]
Equation for potentials in Lorentz gauge

\[ (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \vec{A} = -\frac{4\pi}{c} \vec{J} \]

\[ (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \phi = -4\pi \rho \]

\[ \frac{\partial^2 \phi}{\partial x_i^2} = (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \zeta \text{ Lorentz invariant operator} \]

\[ 4\text{-potential} \quad A_\mu = (\vec{A}, \ i \phi) \]

\[ (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) A_\mu = \frac{-4\pi}{c} \vec{J}_\mu = \frac{\partial^2 A_\mu}{\partial x^2} \]

Lorentz gauge condition is

\[ \nabla \cdot \vec{A} + \frac{\partial \phi}{\partial t} = \frac{\partial A_\mu}{\partial x^\mu} = 0 \]

Electric and magnetic fields

\[ B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \quad \varepsilon, j, k \text{ cyclic permutation of } 1, 2, 3 \]

\[ E_i = -\frac{\partial \phi}{\partial x_i} - \frac{\partial A_i}{\partial t} = c \left( \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} \right) \]

Define field stress tensor

\[ F_{\mu \nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = -F_{\nu \mu} \]

\[ \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & -iE_2 & iE_3 & 0 \end{pmatrix} \]

“Curl” of a 4-vector is a 4x4 anti-symmetric 2nd rank tensor
Inhomogeneous Maxwell's equations can be written in the form:

\[
\frac{\partial F_{\mu\nu}}{\partial x^\mu} = \frac{4\pi}{c^2} j_{\nu} \quad \Rightarrow \quad \begin{cases} 
\nabla \cdot E = 4\pi \rho \\
\nabla \times B - \frac{1}{c^2} \frac{\partial E}{\partial t} = \frac{4\pi}{c} j
\end{cases}
\]

\[
\frac{\partial^2 A_{\mu}}{\partial x^\mu \partial x^\nu} = \frac{4\pi}{c^2} f_{\mu} \quad \text{agrees with previous equation for } A_{\mu}
\]

Transformation law for 2nd rank tensor \( F_{\mu\nu} \):

\[
F'_{\mu\nu} = \frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial x^\nu}{\partial x'^{\nu'}} \quad \text{use } A_{\mu'} = A_{\mu} \frac{\partial x'^{\mu'}}{\partial x^\nu} \\
= \alpha_{\nu\lambda} \alpha_{\mu\sigma} \frac{\partial A_{\lambda}}{\partial x^\nu} - \alpha_{\mu\sigma} \alpha_{\nu\lambda} \frac{\partial A_{\sigma}}{\partial x^\lambda}
\]

\[
\frac{\partial A_{\mu}}{\partial x^\nu} = \alpha_{\mu\sigma} a_{\nu\lambda} F_{\sigma\lambda} \quad \text{lets one find } E' \text{ and } B'
\]

For \( n \)th rank tensor

\[
T_{\mu_1 \mu_2 \cdots \mu_n} = a_{\mu_1 \nu_1} a_{\mu_2 \nu_2} \cdots a_{\mu_n \nu_n} T_{\nu_1 \nu_2 \cdots \nu_n}
\]
\[ \frac{\partial F_{\mu \nu}}{\partial x^\lambda} \text{ is a 4-vector: proof:} \]

\[ \frac{\partial F_{\mu \nu}}{\partial x^\lambda} = a_{\mu \sigma} a_{\nu \lambda} a_{\alpha \nu} \frac{\partial F_{\sigma \lambda}}{\partial x^\gamma} \]

but \( a_{\mu \lambda} = a_{\lambda \mu} \) since inverse = transpose

\( a_{\mu \alpha} a_{\nu \lambda} = a_{\lambda \mu} a_{\nu \alpha} = \delta_{\mu \lambda} \)

\[ \frac{\partial F_{\mu \nu}}{\partial x^\lambda} = a_{\mu \sigma} \frac{\partial F_{\sigma \lambda}}{\partial x^\gamma} = a_{\mu \sigma} \frac{\partial F_{\sigma \lambda}}{\partial x^\gamma} \text{ transforms like 4-vector} \]

To write the homogeneous Maxwell Equations

Construct 3rd rank co-variant tensor

\[ G_{\mu \nu \lambda} = \frac{\partial F_{\mu \nu}}{\partial x^\lambda} + \frac{\partial F_{\nu \mu}}{\partial x^\lambda} + \frac{\partial F_{\lambda \mu}}{\partial x^\nu} \]

transforms as \( G_{\mu \nu \lambda} = a_{\mu \alpha} a_{\nu \beta} a_{\lambda \gamma} G_{\alpha \beta \gamma} \)

in principle \( G \) has \( 4^3 = 64 \) components

But can show that \( G \) is antisymmetric in exchange of any two indices

\[ G_{\nu \mu \lambda} = \frac{\partial F_{\nu \mu}}{\partial x^\lambda} + \frac{\partial F_{\mu \nu}}{\partial x^\lambda} + \frac{\partial F_{\lambda \mu}}{\partial x^\nu} \]

\[ = -\frac{\partial F_{\mu \nu}}{\partial x^\lambda} - \frac{\partial F_{\nu \lambda}}{\partial x^\mu} - \frac{\partial F_{\lambda \mu}}{\partial x^\nu} \text{ as } F \text{ antisymmetric} \]

\[ = -G_{\mu \nu \lambda} \]
Also \( G_{\mu\nu\lambda} = 0 \) if any two indices are equal

\[ \Rightarrow \text{only 4 independent components} \]

\[ G_{012}, G_{013}, G_{023}, G_{123} \]

all other components either vanish or are \( \pm \) one of the above.

The 4 homogeneous Maxwell Equations:

\[ \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \]

can be written as

\[ G_{\mu\nu\lambda} = 0 \]

to see, substitute in definition of \( G \) the definition of \( F \)

\[ G_{\mu\nu\lambda} = \frac{\partial^2 A_\nu}{\partial x_\lambda \partial x_\mu} - \frac{\partial^2 A_\mu}{\partial x_\lambda \partial x_\nu} + \frac{\partial^2 A_\mu}{\partial x_\nu \partial x_\lambda} - \frac{\partial^2 A_\lambda}{\partial x_\nu \partial x_\mu} + \frac{\partial^2 A_\lambda}{\partial x_\mu \partial x_\nu} - \frac{\partial^2 A_\mu}{\partial x_\mu \partial x_\nu} \]

all terms cancel in pairs

\[ = 0 \]

\[ G_{123} = 0 \ \Rightarrow \ \nabla \cdot \mathbf{B} = 0 \]

\[ G_{012} = -\epsilon [\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{c \partial t}]_{\mathbf{z}} = 0 \]

\( \mathbf{3} \) component Faraday's

law
Another way to write homogeneous Maxwell Equation's

Define $\epsilon_{\mu \nu \lambda \sigma} = \begin{cases} +1 & \text{if } \mu \nu \lambda \sigma \text{ is even permutation of } 1234 \\ -1 & \text{if } \mu \nu \lambda \sigma \text{ is odd permutation of } 1234 \\ 0 & \text{otherwise} \end{cases}$

4-d Levi-Civita symbol $\epsilon_{\mu \nu \lambda \sigma}$

Define $\tilde{F}_{\mu \nu} = \frac{i}{2} \epsilon_{\mu \nu \lambda \sigma} F_{\lambda \sigma}$

$\tilde{F}_{\mu \nu} = \begin{pmatrix} 0 & -E_3 & E_2 & iB_1 \\ E_3 & 0 & -E_1 & -iB_2 \\ -E_2 & E_1 & 0 & -iB_3 \\ iB_1 & iB_2 & iB_3 & 0 \end{pmatrix}$

$\delta \tilde{F}_{\mu \nu} = 0$ gives homogeneous Maxwell equations

$\frac{1}{2} F_{\mu \nu} F_{\mu \nu} = B^2 - E^2$ for Lorentz invariant scalars

$-\frac{1}{4} F_{\mu \nu} \tilde{F}_{\mu \nu} = B \cdot E$
From $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we can set

Lorentz transform for $E_i$ and $B_i$

For a transformation from $K$ to $K'$ with $K'$ moving with $\nu$ along $x$, with respect to $K$,

$$E'_1 = E_1, \quad B'_1 = B_1,$$

$$E'_2 = \gamma (E_2 - \frac{\nu}{c} B_3), \quad B'_2 = \gamma (B_2 + \frac{\nu}{c} E_3),$$

$$E'_3 = \gamma (E_3 + \frac{\nu}{c} B_2), \quad B'_3 = \gamma (B_3 - \frac{\nu}{c} E_2).$$

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**Kinematics**

"dot" is \( \frac{d}{ds} \)

**4-momentum**

$$p_\mu = m x_\mu = m u_\mu = (m \gamma \vec{u}, \gamma mc)$$

$$p_\mu^2 = m^2 u_\mu^2 = -m^2 c^2$$

**4-force**

$$K_\mu = (\vec{K}, iK_0) \quad "\text{Minkowski force}"$$

**Newton's 2nd law:**

$$m \frac{d^2 x_\mu}{d s^2} = K_\mu$$

$$\Rightarrow m \frac{d u_\mu}{d s} = \frac{d p_\mu}{d s} = K_\mu$$

$$p_\mu^2 = -m^2 c^2 \quad \Rightarrow \frac{d}{d s} (p_\mu^2) = p_\mu \frac{d p_\mu}{d s} = p_\mu K_\mu = 0$$

$$\Rightarrow m \gamma \vec{u} \cdot \vec{K} - mc \vec{K}_0 = 0 \quad \text{or} \quad K_0 = \frac{\vec{u}}{c} \cdot \vec{K}$$
Define the linear 3-force by
\[ \frac{dp}{dt} = F \]
and
\[ \frac{d^2 p}{ds^2} = \gamma \frac{dp}{dt} = \gamma F \]  \Rightarrow \quad \dot{p} = \gamma F \quad \therefore \quad K_0 = \gamma \frac{\dot{p}^2}{c^2} \cdot F

Consider 4-component of Newton's Law:

\[ \frac{d}{ds} (mc) = iK_0 = i \gamma \frac{\dot{p}^2}{c^2} \cdot \hat{\dot{F}} \]

\[ d(m\gamma c^2) = d \gamma \frac{\dot{p}^2}{c^2} \cdot \hat{\dot{F}} = d \gamma \frac{\dot{p}^2}{c^2} \cdot F = d \gamma \frac{\dot{p}^2}{c^2} \frac{\dot{F}}{c^2} \]

Work-energy theorem: \( d(m\gamma c^2) = d \gamma \frac{\dot{p}^2}{c^2} \cdot \hat{\dot{F}} = \text{work done} \)

\[ \Rightarrow \quad d(m\gamma c^2) \text{ is change in kinetic energy} \]

\[ E = m\gamma c^2 \text{ is relativistic energy} \]

\[ \hat{p}_\mu = (\vec{p}, \frac{iE}{c}) \quad \hat{p} = m\gamma \frac{\vec{p}}{c} \quad \frac{E}{c} = m\gamma c^2 \]

\[ \begin{align*}
E &= mc^2 \\
&= \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \\
&= mc^2 \left(1 + \frac{1}{2} \frac{v^2}{c^2}\right) = mc^2 + \frac{1}{2}mv^2
\end{align*} \]

\[ \text{small } \frac{v}{c} \quad \text{rest mass} \quad \text{energy} \]

\[ \text{non-rel} \quad \text{kinetic energy} \]

\[ \frac{d\hat{p}_\mu}{ds} = \kappa \mu \quad \text{Therefore} \]

\[ \text{relativistic analog of Newton's 3rd law} \]

\[ \text{as well as law of conservation of energy} \]
Lorentz Force

\[ \frac{dp}{ds} = Kp \]

What is the \( Kp \) that represents the Lorentz force and how can we write it in a covariant way?

\( Kp \) should depend on the fields \( F_{\mu \nu} \) and the particle's trajectory \( x_\mu \)

\[ as \quad s \to 0 \quad K = qE \]

\( Kp \) cannot depend directly on \( x_\mu \) as it should be independent of origin of coordinates. So can depend only on \( x_\mu, x_{\nu}, \) etc.

\[ as \quad s \to 0, \quad K \text{ does not depend on the acceleration, so } K \text{ does not depend on } x_\mu \]

\( Kp \) only depends on \( F_{\mu \nu} \) and \( x_\mu \)

We need to form a \( 4\)-vector out of \( F_{\mu \nu} \) and \( x_\mu \) that is linear in the fields \( F_{\mu \nu} \) and proportional to the charge \( q \).

The only possibility is

\[ q f \left( x_\mu^2 \right) F_{\mu \nu} x_\nu \]
But \( x_\mu = -ct \) is a constant. Choose \( f(x_\mu) = \frac{1}{c} \)

\[
K_\mu = \frac{q}{c} F_{\mu\nu} \dot{x}_\nu \quad \text{is only possibility}
\]

This gives force

\[
\vec{F} = \frac{1}{8} \vec{K}
\]

\[
F_i = \frac{1}{8} K_i = \frac{q}{yc} \left( F_{ij} \dot{x}_j + F_{i4} \dot{x}_4 \right)
\]

\[
= \frac{\varepsilon}{yc} \left( \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) \dot{x}_j + \frac{\varepsilon}{yc} \left( -i E_i (ic \dot{y}) \right)
\]

\[
= \frac{\varepsilon}{yc} \left[ E_{ijk} B_k \dot{y} \dot{y}_j \right] + \frac{\varepsilon}{yc} \dot{E}_i c \dot{y}
\]

\[
= \frac{q}{\varepsilon} E_i + \frac{q}{\varepsilon} E_{ijk} \frac{\dot{x}_i}{c} B_k
\]

\[
\vec{F} = \frac{q}{\varepsilon} \vec{E} + \frac{q}{\varepsilon} \frac{\dot{x}_i}{c} \vec{B}
\]

Lorentz force is the same form in all inertial frames. No relativistic modification is needed.
Relativistic Larmor's formula

\[ P = \frac{z}{3} \frac{\xi^2 [\alpha \beta \gamma]}{c^3} \]

Consider inertial frame in which charge is instantaneously at rest. Call this rest frame \( K \).

Power radiated in \( K \) is

\[ P = \frac{d \hat{E}}{dt} \]

where \( \hat{E} \) is energy radiated. In \( K' \), the momentum density \( \hat{P} = \frac{1}{4\pi c} \hat{E} \times \hat{B} \sim \hat{F} \) is in outward radial direction. Integrating over all directions, the radiated momentum vanishes \( \hat{P} = 0 \).

Energy-momentum is a 4-vector \( (\hat{P}, \xi \hat{c}) \).

To get radiated energy in original frame \( K \) we can use Lorentz transform

\[
\xi = \gamma (\hat{\xi} \frac{\hat{c}}{c} - \frac{\hat{\xi}}{c}) \quad \Rightarrow \quad \xi = \gamma \xi^0 \quad \text{as} \quad \hat{\xi} = 0
\]

\[ \frac{dE}{dt} = \gamma \frac{dE^0}{dt} \quad \text{in time interval in} \ K
\]

(\( d\hat{r}^0 = 0 \) as charge stays at origin in \( K \)).

So

\[ \frac{dE}{dt} = \gamma \frac{dE^0}{dt} = \frac{dE^0}{dt} \quad \Rightarrow \quad P = \hat{P} \]

radiated power in Lorentz invariant!
in $\mathbb{R}$ we can use non-relativistic Larmor's formula since $v=0$. So

$$P = \frac{2}{3} \frac{q^2 \mathbf{a}^2}{c^3} \quad \text{a \cdot \text{acceleration in } \mathbb{R}}$$

To write an expression without explicitly making mention of frame $\mathbb{R}$, we need to find a Lorentz invariant scalar that reduces to $\mathbf{a}^2$ as $v \to 0$.

Only choice is $\gamma^2 \mathbf{a}^2$ the 4-acceleration $\gamma^2 \mathbf{a} = \frac{d\gamma}{ds}$

$$\gamma^2 \mathbf{a}_\mu = \gamma \frac{d\gamma}{ds} = \gamma \frac{d}{dt} \left( \gamma \mathbf{u} \cdot \mathbf{e}_\gamma \right)$$

$$\gamma = \gamma^2 \frac{d\mathbf{u}}{dt} + \gamma \mathbf{u} \cdot \frac{d\gamma}{dt}$$

$$\gamma^2 \mathbf{a}_\mu = \gamma \mathbf{u} \cdot \frac{d\gamma}{dt}$$

$$\frac{d\gamma}{dt} = \frac{1}{\gamma} \frac{d}{dt} \left( \frac{1}{\sqrt{1 - v^2/c^2}} \right) = \frac{\frac{v}{c^2} \frac{d\mathbf{u}}{dt}}{(1 - v^2/c^2)^{3/2}} = \frac{v}{c^2} \frac{d\mathbf{u}}{dt}$$

as $v \to 0$, $\gamma \to 1$, $\frac{d\gamma}{dt} \to 0$, so

$$\mathbf{a} \to \frac{d\mathbf{u}}{dt} = \mathbf{a}$$

$$\mathbf{a}_\mu \to |\mathbf{a}|^2 \text{ as desired}$$

Relative Larmor formula

$$P = \frac{2}{3} \frac{q^2 \mathbf{a}^2}{c^3} \quad \text{a \cdot \text{acceleration in } \mathbb{R}}$$
\[ \alpha_\mu = \left( \gamma^2 \frac{d\mathbf{u}}{dt} + \gamma \mathbf{v} \frac{d\mathbf{y}}{dt} - c \mathbf{v} \frac{d\mathbf{r}}{dt} \right) \]

\[ \frac{d\mathbf{v}}{dt} = \frac{1}{c^2} \mathbf{\hat{v}} \cdot \mathbf{a} \]

\[ \alpha_\mu = \left( \gamma^2 \mathbf{a} + \frac{\gamma^4}{c^2} \left( \mathbf{v} \cdot \mathbf{a} \right) \mathbf{v} + \frac{c}{c^2} \gamma^4 \mathbf{\hat{v}} \cdot \mathbf{a} \right) \]

\[ \alpha_\mu = \gamma^4 \mathbf{a}^2 + \gamma^4 \frac{\left( \mathbf{\hat{v}} \cdot \mathbf{a} \right)^2 \mathbf{v}^2 + 2 \gamma^6 \left( \mathbf{\hat{v}} \cdot \mathbf{a} \right)^2}{c^2} - \frac{\gamma^8 \left( \mathbf{\hat{v}} \cdot \mathbf{a} \right)^2}{c^2} \]

\[ = \gamma^4 \left[ a^2 + \gamma^4 \frac{\left( \mathbf{\hat{v}} \cdot \mathbf{a} \right)^2}{c^2} \left( \frac{\mathbf{v}^2}{c^2} - 1 \right) + \frac{2 \gamma^6 \left( \mathbf{\hat{v}} \cdot \mathbf{a} \right)^2}{c^2} \right] \]

\[ = \gamma^4 \left[ a^2 - \gamma^2 \left( \mathbf{\hat{v}} \cdot \mathbf{a} \right)^2 + \frac{2 \gamma^2 \left( \mathbf{\hat{v}} \cdot \mathbf{a} \right)^2}{c^2} \right] \]

\[ \alpha_\mu = \gamma^4 \left[ a^2 + \gamma^2 \left( \mathbf{\hat{v}} \cdot \mathbf{a} \right)^2 \right] \]

As \( \mathbf{\hat{v}} \to 0 \), \( \alpha_\mu \to a^2 \)

\[ \alpha_\mu^2 = \mathbf{\hat{a}}^2 \text{ Lorentz invariant} \]

\[ \mathbf{\hat{a}} = \text{acceleration in instantaneous rest frame} \]

For a charge accelerating in linear motion, \( \left( \mathbf{\hat{v}} \cdot \mathbf{a} \right)^2 = v^2 a^2 \)

\[ \alpha_\mu^2 = \gamma^4 a^2 \left( 1 + \gamma^2 v^2 \right) = \gamma^6 a^2 \]

\[ P = \frac{2}{3} c^3 \gamma^6 \]

For a charge in circular motion \( \left( \mathbf{\hat{v}} \cdot \mathbf{a} \right) = 0 \)

\[ \alpha_\mu^2 = \gamma^4 a^2 \]

\[ P = \frac{2}{3} c^3 \gamma^4 \]