PHY 415 **Final Exam** **Fall 2017**

1) [30 points total]

Consider a dielectric sphere of radius R with dielectric constant ε. At the center of the sphere is a point charge q. The sphere is placed in a uniform external electric field $\mathbf{E}_0 = E_0\hat{\mathbf{z}}$.

a) [20 pts] What is the electrostatic potential inside and outside the sphere?

b) [10 pts] What is the total surface charge density induced on the surface of the sphere?

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2) [35 points total]

Consider a bound atomic electron of charge q in a dielectric medium with a complex, frequency dependent, atomic polarizability $\alpha(\omega)$. We can write $\alpha = \alpha_1 + i\alpha_2$, where α_1 and α_2 are the real and imaginary parts of α respectively. The dielectric function may be taken to be $\varepsilon(\omega) = 1 + 4\pi N\alpha(\omega)$ where N is the density of polarizable atoms. We take $\mu = 1$ for this medium. Note: for this problem you should not assume any particular functional form for $\alpha(\omega)$.

An oscillating electric field $\mathbf{E}(t) = \mathrm{Re}\left[\mathbf{E}_\omega \mathrm{e}^{i(\delta-\omega t)}\right]$ exerts a force on the bound electron cloud. Assume $\mathbf{E}_\omega$ is a real valued vector and δ is an arbitrary phase factor.

a) [5 pts] The resulting velocity of the electron cloud center of mass can be written as $\mathbf{v}(t) = \mathrm{Re}\left[\mathbf{v}_\omega \mathrm{e}^{i(\delta-\omega t)}\right]$. Find $\mathbf{v}_\omega$ in terms of $\mathbf{E}_\omega$ and $\alpha(\omega)$.

b) [10 pts] If W is the work done on the electron by the electric field in one period of oscillation T, then find the average work done over one period, i.e. W/T, where $T = 2\pi/\omega$. This is the energy absorbed by the electron. Express your answer in terms of $\mathbf{E}_\omega$ and $\alpha(\omega)$.

c) [10 pts] Consider a linearly polarized electromagnetic plane wave traveling in the $+\hat{\mathbf{z}}$ direction. The electric field of this wave can be written as,

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\left[\mathbf{E}_\omega \mathrm{e}^{-k_2 z}\mathrm{e}^{i(k_1 z-\omega t)}\right] = \mathbf{E}_\omega \mathrm{e}^{-k_2 z}\cos(k_1 z - \omega t),$$

where the complex wavenumber $k = k_1 + ik_2$ is related to the complex dielectric function $\varepsilon = \varepsilon_1 + i\varepsilon_2$ by $k^2 = (\omega^2/c^2)\varepsilon(\omega)$.

Assume the material has a cross-sectional area A in the xy plane. Using your results from (b), find the total energy per unit cross-sectional area, per period of oscillation, that is absorbed by the medium in the half space $z > 0$. Express your answer in terms of $\mathbf{E}_\omega$, ε, ω and k.

d) [10 pts] The magnetic field of this electromagnetic wave can be written as

$$\mathbf{B}(\mathbf{r}, t) = \frac{c|k|}{\omega}(\hat{\mathbf{z}} \times \mathbf{E}_\omega)\mathrm{e}^{-k_2 z}\cos(k_1 z - \omega t + \varphi)$$

where $|k| = \sqrt{k_1^2 + k_2^2}$ and the phase shift $\varphi = \tan^{-1}(k_2/k_1)$. Taking the Poynting vector for this wave as $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$, show that the total energy per unit cross-sectional area, per period of oscillation, which flows through the plane at $z = 0$ is just equal to the energy absorbed by the half-space of the material at $z > 0$, as computed in part (c). This is as it must be, since the energy dissipated in the half-space $z > 0$ must equal the energy flowing into the half-space through the plane at $z = 0$.

3) [35 points total]

An insulating thin circular ring of radius R lies centered about the origin in the xy plane at $z = 0$. It carries a line charge density $\lambda(\varphi) = \lambda_0 \sin\varphi$, where λ_0 is a constant and φ is the usual polar angle. The ring is now set spinning counterclockwise at constant angular velocity ω about the $\hat{z}$ axis.

a) [5 pts] Compute the line charge density of the spinning ring, $\lambda(\varphi, t)$.

b) [10 pts] Compute the electric dipole moment $\mathbf{p}(t)$ of this spinning ring. Then write it in the form $\mathbf{p}(t) = \mathrm{Re}\left[\mathbf{p}_\omega \mathrm{e}^{-i\omega t}\right]$ to find $\mathbf{p}_\omega$.

Working in the electric dipole approximation, in the radiation zone limit:

c) [10 pts] Using the result from (b), compute the time averaged Poynting vector $\langle \mathbf{S}(\mathbf{r}) \rangle$.

d) [5 pts] Compute the cross-section of the radiated power $dP/d\Omega$, and sketch a polar plot of $dP/d\Omega$ as a function of the spherical angle θ.

e) [5 pts] Compute the total radiated power P.

Note, the electric dipole approximation for the electric and magnetic fields in the radiation zone gives,

$$\begin{aligned}
\mathbf{E}(\mathbf{r}, t) &= \mathrm{Re}\left[-k^2 \frac{\mathrm{e}^{i(kr-\omega t)}}{r} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{p}_\omega)\right] \\
\mathbf{B}(\mathbf{r}, t) &= \mathrm{Re}\left[k^2 \frac{\mathrm{e}^{i(kr-\omega t)}}{r} \hat{\mathbf{r}} \times \mathbf{p}_\omega\right]
\end{aligned}$$

where $k = \omega/c$. Hint: when taking the real part of the above expressions, it may be helpful to write $\mathbf{p}_\omega = \mathbf{p}_1 + i\mathbf{p}_2$, where $\mathbf{p}_1$ and $\mathbf{p}_2$ are the real and imaginary parts of $\mathbf{p}_\omega$, then work out the calculation in terms of $\mathbf{p}_1$ and $\mathbf{p}_2$ before substituting in the values for these quantities that you found in part (b).