

Helmholtz Theorem

Suppose $\vec{\nabla} \cdot \vec{E}(\vec{r}) = f(\vec{r})$ } for \vec{r} in a volume V
 $\vec{\nabla} \times \vec{E}(\vec{r}) = \vec{g}(\vec{r})$ }
 $\vec{E}(\vec{r}) = \vec{h}(\vec{r})$ for \vec{r} on surface S of vol V

Then if we know $f(\vec{r})$, $\vec{g}(\vec{r})$ and $\vec{h}(\vec{r})$, that information uniquely determines the vector function $\vec{E}(\vec{r})$

Proof:

Suppose we had two different solutions $\vec{E}(\vec{r})$ and $\vec{E}'(\vec{r})$
Then define

$$\vec{G}(\vec{r}) = \vec{E}(\vec{r}) - \vec{E}'(\vec{r})$$

\vec{G} must satisfy

$$\left. \begin{aligned} \vec{\nabla} \cdot \vec{G} &= 0 \\ \vec{\nabla} \times \vec{G} &= 0 \end{aligned} \right\} \text{ for all } \vec{r} \text{ in } V$$

$$\vec{G} = 0 \quad \text{for all } \vec{r} \text{ on } S$$

Now $\vec{\nabla} \times \vec{G} = 0$ implies we can find a scalar function ϕ such that $\vec{G} = \vec{\nabla} \phi$. Then
 $\vec{\nabla} \cdot \vec{G} = 0 \Rightarrow \nabla^2 \phi = 0$ for all \vec{r} in V .

A function ϕ that satisfies $\nabla^2 \phi = 0$ within a region V is said to be a harmonic function on V .

An important property of harmonic functions is that the value at a position \vec{r} , is equal to the average of the values on the surface of a sphere centered at \vec{r} .

$$\phi(\vec{r}) = \frac{1}{4\pi R^2} \oint_{\Sigma} \phi(\vec{r}') d\vec{a}'$$

Σ surface of sphere of radius R centered at \vec{r} .

From this property we can conclude that a harmonic function on V can have no local maximum or minimum within the volume V . All maxima and minima must lie on surface S of V .

Proof: Just consider a small sphere centered on \vec{r} that fits within the volume V . If \vec{r} was a max, then for \vec{r}' on surface of sphere, $\phi(\vec{r}') < \phi(\vec{r})$. But then we would have $\phi(\vec{r}) < \frac{1}{4\pi R^2} \oint da' \phi(\vec{r}')$ in violation of the above property of harmonic functions.

Back to our function $\vec{G}(\vec{r})$. We have

$$\vec{\nabla} \cdot \vec{G} = 0, \quad \vec{G} = \vec{\nabla} \phi \Rightarrow \nabla^2 \phi = 0 \text{ in } V$$

$$\vec{G} = \vec{\nabla} \phi = 0 \text{ on surface } S \text{ of } V \Rightarrow \phi = \text{constant on } S.$$

All max and min of ϕ must be on surface S

$$\Rightarrow \phi_{\max} = \phi_{\min} = \text{constant},$$

$$\Rightarrow \phi = \text{constant throughout volume } V$$

$$\Rightarrow \vec{\nabla} \phi = \vec{G} = 0 \text{ throughout } V$$

$$\Rightarrow \vec{E} = \vec{E}' \text{ for all } \vec{r} \text{ in } V$$

\Rightarrow solution is unique!

Magnetostatics

Lorentz Force

a charge q , in motion with velocity \vec{v} , feels the force

$$\vec{F} = q (\vec{E} + k_4 \vec{v} \times \vec{B}) \quad \leftarrow \text{Lorentz force}$$

\vec{B} is the magnetic field at the position of the charge.
 k_4 is a universal constant.

Just as the constant k_1 fixed the units of charge q , the constant k_4 can be viewed as fixing the units of B magnetic field. By choosing the units of q and B appropriately, we are free to choose any values for k_1 and k_4 .

Magnetic field \vec{B} is generated by moving charge.
A charge q' with velocity \vec{v}' ($v' \ll c$) located at the origin $\vec{r}'=0$ produces a magnetic field at position \vec{r} ,

holds only non relativistically $\vec{B}(\vec{r}) = k_5 \frac{q' \vec{v}' \times \vec{r}}{r^3} = \frac{k_5}{k_1} \vec{v}' \times \vec{E}(\vec{r})$

k_5 is a universal constant. we will see that it cannot be chosen independently of k_1 and k_4 .
(since k_1 fixed units of q , and k_4 fixed units of \vec{B} , there are no further new quantities whose units could be adjoined to allow us to fix k_5 arbitrarily)

(11)

The force on a charge q at position \vec{r} , moving with velocity \vec{v} , due to a charge q' at the origin moving with velocity \vec{v}' is, in non-relativistic limit ($v, v' \ll c$),

$$\vec{F} = k_1 q q' \frac{\vec{r}}{r^3} + k_4 k_5 q q' \frac{\vec{v} \times (\vec{v}' \times \vec{r})}{r^3}$$

↑
Coulomb force

↑
magnetic analog of Coulomb force

The magnetic part is just the point charge equivalent of the Biot-Savart law for the force between current carrying wires. If we regard $q\vec{v} = \vec{I}$ as the current of charge q , and $q'\vec{v}' = \vec{I}'$ as the current of charge q' , then the magnetic force is $k_4 k_5 \frac{\vec{I} \times (\vec{I}' \times \frac{\vec{r}}{r^3})}{r^3}$ which is the Biot-Savart Law.

Rewrite above force as

$$\vec{F} = k_1 \left(1 + \frac{k_4 k_5}{k_1} \vec{v} \times \vec{v}' \times \right) \frac{\vec{r}}{r^3} q q'$$

we see that $\left(\frac{k_4 k_5}{k_1} \right)$ has units of $(\text{velocity})^{-2}$

it must be independent of whatever convention one used to choose the units of q or B (ie independent of choices for k_1 and k_4). Experimentally it is found that

$$\left(\frac{k_4 k_5}{k_1} \right) = \frac{1}{c^2}$$

c = speed of light in vacuum

Continuum current density

For charges q_i at positions $\vec{r}_i(t)$ with $\vec{v}_i = \frac{d\vec{r}_i}{dt}$
we define the current density

$$\vec{j}(\vec{r}, t) = \sum_i q_i \vec{v}_i(t) \delta(\vec{r} - \vec{r}_i(t))$$

units of \vec{j} are $(\text{charge}) \left(\frac{\text{length}}{\text{time}} \right) \left(\frac{1}{\text{length}^3} \right) = \left(\frac{\text{charge}}{\text{area} \cdot \text{time}} \right)$

charge per unit area per unit time

For a surface S'

$$\int_{S'} da \hat{n} \cdot \vec{j} = I \quad \text{current (charge per unit time)} \\ \text{passing through surface } S'$$

Charge Conservation

vol V bounded by surface S'

$$\frac{d}{dt} \int_V d^3r \rho(\vec{r}, t) = - \oint_{S'} da \hat{n} \cdot \vec{j}$$

rate of change of total charge in V = (-) charge flowing out of V through S' per unit time

$$\text{use } \oint_{S'} da \hat{n} \cdot \vec{j} = \int_V d^3r \vec{\nabla} \cdot \vec{j} = - \int_V d^3r \frac{\partial \rho(\vec{r}, t)}{\partial t}$$

\Rightarrow local charge conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

A static situation has $\frac{\partial f}{\partial t} = 0$

\Rightarrow magnetostatics is defined by the condition $\vec{\nabla} \cdot \vec{j} = 0$

Differential formulation of Biot-Savart

For a set of charges q_i at \vec{r}_i we have

$$\vec{B}(\vec{r}) = \sum_i k_S q_i \vec{v}_i \times \frac{(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}$$

$$= k_S \int d^3r' \vec{j}(\vec{r}') \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$= k_S \int d^3r' \vec{j}(\vec{r}') \times \vec{\nabla} \left(\frac{-1}{|\vec{r} - \vec{r}'|} \right)$$

$$\vec{B}(\vec{r}) = k_S \vec{\nabla} \times \left[\int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right]$$

where we used $\vec{\nabla} \times (\vec{A} \phi) = -\vec{A} \times \vec{\nabla} \phi$ when \vec{A} is indep of \vec{r}

$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$ since $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ for any vector function \vec{A}
integral form $\oint d\vec{a} \cdot \vec{B} = 0$

$$\vec{\nabla} \times \vec{B} = k_S \vec{\nabla} \times \left[\vec{\nabla} \times \left(\int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) \right]$$

use $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

$$\vec{\nabla} \times \vec{B} = k_5 \vec{\nabla} \left[\int d^3r' \frac{\vec{\nabla} \cdot (\vec{j}(\vec{r}'))}{|\vec{r} - \vec{r}'|} \right]$$

$$- k_5 \int d^3r' \vec{j}(\vec{r}') \nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right)$$

in the 2nd term, $\nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = -4\pi \delta(\vec{r} - \vec{r}')$

in the 1st term, $\frac{\vec{\nabla} \cdot \vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} = \vec{j}(\vec{r}') \cdot \vec{\nabla} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right)$

$$= -\vec{j}(\vec{r}') \cdot \vec{\nabla}' \left(\frac{1}{|\vec{r} - \vec{r}'|} \right)$$

since $\vec{\nabla} = -\vec{\nabla}'$

So $\int d^3r' \frac{\vec{\nabla} \cdot (\vec{j}(\vec{r}'))}{|\vec{r} - \vec{r}'|} = - \int d^3r' \vec{j}(\vec{r}') \cdot \vec{\nabla}' \left(\frac{1}{|\vec{r} - \vec{r}'|} \right)$

integrate by parts $= \int d^3r' (\vec{\nabla}' \cdot \vec{j}(\vec{r}')) \left(\frac{1}{|\vec{r} - \vec{r}'|} \right)$

surface term $\rightarrow 0$ as

we take surface $\rightarrow \infty$

since $\vec{j} \rightarrow 0$ as $r \rightarrow \infty$

But for magnetostatics $\vec{\nabla} \cdot \vec{j} = 0 \Rightarrow$ only 2nd term remains

Thus, for magnetostatics

$$\vec{\nabla} \times \vec{B} = 4\pi k_5 \vec{j} \quad \text{Ampere's law}$$

integral form $\oint_C d\vec{l} \cdot \vec{B} = 4\pi k_5 \int_S da \hat{n} \cdot \vec{j}$

\uparrow curve bounding surface \uparrow

Although above diff eqs were derived starting from a "non-relativistic"

point-charge Biot-Savart law? They actually remain true for all magnetostatic situations

So far electrostatics $\begin{cases} \vec{\nabla} \cdot \vec{E} = 4\pi k_1 \rho & \text{Gauss} \\ \vec{\nabla} \times \vec{E} = 0 \end{cases}$

magnetostatics $\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = 4\pi k_5 \vec{j} & \text{Ampere} \end{cases}$

charge conservation $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$

Time Dependent situations

Faraday's law of induction $\vec{\nabla} \times \vec{E} \neq 0!$ mag flux

$$\oint_C d\vec{l} \cdot \vec{E} = -k_3 \frac{\partial}{\partial t} \int_S da \hat{n} \cdot \vec{B}$$

$\begin{matrix} \nearrow \text{emf around loop} \\ \searrow \text{mag flux} \end{matrix}$ $\mathcal{E}_{\text{mf}} = -k_3 \frac{d\Phi}{dt}$

choose \hat{n} voltage around closed loop \sim time rate of change of magnetic flux through loop
 and $d\vec{l}$ consistent with Right Hand Rule

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -k_3 \frac{\partial \vec{B}}{\partial t}} \quad k_3 \text{ is universal constant}$$

Maxwell correction to Ampere's law

In our derivation of $\vec{\nabla} \times \vec{B} = 4\pi k_5 \vec{j}$ we used $\vec{\nabla} \cdot \vec{j} = 0$. This is only true for magnetostatics - it is NOT true in general

Alternatively, since $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$ always, if Ampere's law was true, we would necessarily conclude that $\vec{\nabla} \cdot \vec{j} = 0$. But $\vec{\nabla} \cdot \vec{j} = -\frac{\partial \rho}{\partial t} \neq 0$ in general.