proof that we can always find $\vec{A}$ and ϕ that satisfy the Lorentz gauge condition

Suppose $\vec{\nabla}\times\vec{A} = \vec{B}$ and $-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \vec{E}$

but $\frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla\cdot\vec{A} = D(r,t) \neq 0$

Construct $\vec{A}' = \vec{A} + \vec{\nabla}\chi$

$$\phi' = \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}$$

by gauge invariance we know $\vec{A}'$ and ϕ' give the same $\vec{E}$ and $\vec{B}$ as before.

now: $\vec{\nabla}\cdot\vec{A}' + \frac{1}{c}\frac{\partial \phi'}{\partial t} = \vec{\nabla}\cdot\vec{A} + \nabla^2\chi + \frac{1}{c}\frac{\partial \phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2}$

$$= D - \Box^2\chi$$

So $\vec{A}'$ and ϕ' will be in the Lorentz gauge provided we choose $\chi(\vec{r},t)$ such that

$$\Box^2\chi = D$$ ← inhomogeneous wave equation

Just like there is always a solution to Poisson's Eq $\nabla^2\phi = f$, so there is always a solution to the inhomogeneous wave equation, hence we can always find a $\chi(\vec{r},t)$ that transforms to the Lorentz gauge

Note: Lorentz gauge condition does <u>not</u> uniquely determine $\vec{A}$ and ϕ. If one ~~constructs~~ has $\vec{A}$ and ϕ obeying Lorentz gauge condition, and then constructs

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi$$
$$\phi' = \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}$$

Then $\vec{A}'$ and ϕ' will also be in Lorentz gauge <u>provided</u> $\Box^2 \chi = 0$ (proof left to reader)

2) <u>Coulomb Gauge</u>

<u>gauge constraint</u>: require $\vec{\nabla}\cdot\vec{A} = 0$
if $\vec{A}$ is in the Coulomb Gauge, then
$\vec{A}' = \vec{A} + \vec{\nabla}\chi$ will also be in Coulomb gauge
provided $\nabla^2 \chi = 0$.

Then Gauss' law becomes

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho$$

$$\Rightarrow \boxed{\nabla^2\phi = -4\pi\rho}$$ same as electrostatics!

$$\Rightarrow \phi(\vec{r},t) = \int d^3r' \, \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

no matter what motion the source $\rho(\vec{r},t)$ has!
ϕ is given by the instantaneous Coulomb potential even though electromagnetic fields have a finite velocity of propagation c!

Ampere's Law becomes:

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{j} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)$$

$$\Box^2\vec{A} = \frac{4\pi}{c}\vec{j} - \frac{1}{c}\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) \quad \text{since } \vec{\nabla}\cdot\vec{A} = 0$$

Now use the solution for ϕ in the Coulomb gauge to write

$$\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) = \vec{\nabla}\left[\int d^3r' \, \frac{\partial\rho(\vec{r}',t)}{\partial t}\frac{1}{|\vec{r}-\vec{r}'|}\right]$$

$$= -\vec{\nabla}\left[\int d^3r' \, \frac{\vec{\nabla}'\cdot\vec{j}(\vec{r}',t)}{|\vec{r}-\vec{r}'|}\right]$$

last step follows from conservation of charge $\vec{\nabla}\cdot\vec{j} = -\frac{\partial\rho}{\partial t}$

To see the meaning of this term, recall (and we will soon demonstrate explicitly) that any vector function $\vec{f}(\vec{r},t)$ can always be written as the sum of a curlfree part and a divergenceless part

$$\vec{f} = \vec{f}_{\parallel} + \vec{f}_{\perp} \quad \text{where} \quad \vec{\nabla}\times\vec{f}_{\parallel} = 0 \ \text{curlfree}$$
$$\vec{\nabla}\cdot\vec{f}_{\perp} = 0 \ \text{divergenceless}$$

when $\vec{\nabla}\cdot\vec{f}$ and $\vec{\nabla}\times\vec{f}$ are localized functions that vanish as $\vec{r}\to\infty$, we have for solutions (proof to follow)

$$\vec{f}_{\parallel}(\vec{r}) = -\frac{1}{4\pi}\vec{\nabla}\int d^3r' \, \frac{\vec{\nabla}'\cdot\vec{f}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\vec{f}_{\perp}(\vec{r}) = \frac{1}{4\pi}\vec{\nabla}\times\int d^3r' \, \frac{\vec{\nabla}'\times\vec{f}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

The curlfree part is also called the longitudinal part
the divergenceless part is also called the transverse part

Returning to Ampere's law we see that the term

$$\vec{\nabla}\left(\frac{\partial \phi}{\partial t}\right) = -\vec{\nabla} \int d^3r' \left[\frac{\vec{\nabla}'\cdot\vec{j}(r',t)}{|\vec{r}-\vec{r}'|}\right]$$

$$= 4\pi \vec{j}_{\parallel}(\vec{r},t)$$

So Amperes law becomes

$$\Box^2 \vec{A} = \frac{4\pi}{c}\vec{j} - \frac{4\pi}{c}\vec{j}_{\parallel}$$

$$\boxed{\Box^2 \vec{A} = \frac{4\pi}{c}\vec{j}_{\perp}}$$

In Coulomb gauge, only the transverse part of $\vec{j}$ serves as a source for $\vec{A}$.

$\vec{A}$ describes the transverse modes, ie the EM radiation (recall in EM waves, the fields are always $\perp$ direction of propagation)

ϕ describes the longitudinal modes

Coulomb gauge is not Lorentz invariant - if $\vec{\nabla}\cdot\vec{A} = 0$ in one inertial reference frame, in general $\vec{\nabla}\cdot\vec{A} \neq 0$ in another.

In Coulomb gauge, if $\rho = 0$, then $\phi = 0$ and

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

Transverse + Longitudinal Parts of vector functions

To prove the preceding claim, $\vec{j} = \vec{j}_{\parallel} + \vec{j}_{\perp}$, where $\vec{\nabla}\times\vec{j}_{\parallel} = 0$ and $\vec{\nabla}\cdot\vec{j}_{\perp} = 0$, we first digress to prove Helmholtz Theorem.

Helmholtz Theorem: For a vector function $\vec{f}(\vec{r})$ if one knows the divergence and curl of $\vec{f}$ then one can ~~uniquely~~ uniquely determine $\vec{f}$ itself.

That is, if

$$\vec{\nabla}\cdot\vec{f} = 4\pi D(\vec{r})$$ where $D(\vec{r})$ is a known scalar function

$$\vec{\nabla}\times\vec{f} = 4\pi \vec{C}(\vec{r})$$ where $\vec{C}(\vec{r})$ is a known vector function.

~~Then one can solve for~~

And if well defined boundary conditions on $\vec{f}$ are known (here we will assume $\vec{f}(\vec{r})\to 0$ as $\vec{r}\to\infty$) Then there is a unique solution for $\vec{f}(\vec{r})$.

We prove this by construction!

Assume a solution of the form

$$\vec{f} = -\vec{\nabla}\varphi + \vec{\nabla}\times\vec{W}$$ where φ is a scalar and $\vec{W}$ a vector

Now we show that we can find such a solution

First consider

$$\vec{\nabla}\cdot\vec{f} = -\nabla^2\varphi + \vec{\nabla}\cdot(\vec{\nabla}\times\vec{W}) = -\nabla^2\varphi + 0 = 4\pi D(\vec{r})$$

So $\quad -\nabla^2\varphi = 4\pi D(\vec{r})$ This is just Poisson's eqn we saw in electrostatics

Solution when $\varphi(\vec{r})\to 0$ as $\vec{r}\to\infty$ is given by

$$\boxed{\varphi(\vec{r}) = \int d^3r' \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}$$

Coulomb-like integral solution

Now consider

$$\vec{\nabla}\times\vec{f} = -\vec{\nabla}\times\vec{\nabla}\varphi + \vec{\nabla}\times(\vec{\nabla}\times\vec{W}) = 0 - \nabla^2\vec{W} + \vec{\nabla}(\vec{\nabla}\cdot\vec{W})$$
$$= 4\pi\vec{C}(\vec{r})$$

Choose a gauge in which $\vec{\nabla}\cdot\vec{W}=0$ (just like Coulomb gauge in magnetostatics)

Then $\quad -\nabla^2\vec{W} = 4\pi\vec{C}(\vec{r})$

$$\boxed{\vec{W}(\vec{r}) = \int d^3r' \frac{\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}$$

just like solution for vector pot $\vec{A}$ in magnetostatics

So we have constructed a solution

$$f(\vec{r}) = -\vec{\nabla}\varphi + \vec{\nabla}\times\vec{W}$$

$$= -\vec{\nabla}\int d^3r' \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \vec{\nabla}\times\int d^3r' \frac{\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

where $\vec{\nabla}\cdot\vec{f} = 4\pi D$ and $\vec{\nabla}\times\vec{f} = 4\pi\vec{C}$

Note: For above solution to be well defined, the integrals must converge. They will converge if the "sources" $D(\vec{r})$ and $\vec{C}(\vec{r})$ are sufficiently "localized" in space, ie $D(\vec{r}) \to 0$, $\vec{C}(\vec{r}) \to 0$ sufficiently fast as $\vec{r} \to \infty$.

Now we show that the above solution is unique.

Suppose there was another solution g such that

$$\vec{\nabla} \cdot \vec{g} = 4\pi D \quad \text{and} \quad \vec{\nabla} \times \vec{g} = 4\pi \vec{C}$$

Consider $\vec{h} \equiv \vec{f} - \vec{g}$ then

$$\vec{\nabla} \cdot \vec{h} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{h} = 0$$

Can show that only such $\vec{h}$ that also has $\vec{h}(\vec{r}) \to 0$ as $\vec{r} \to \infty$ is $\vec{h} \equiv 0$, so $\vec{g} = \vec{f}$ and solution is unique.

As a consequence of Helmholtz theorem we have also shown the following

① Any vector function $\vec{f}$ can be written in terms of a scalar and vector potential

$$\vec{f} = -\vec{\nabla}\varphi + \vec{\nabla} \times \vec{W}$$

or equivalently

(2) Any vector function $\vec{f}$ can be written in terms of a curl free and a divergenceless part

$$\vec{f} = \vec{f}_{\parallel} + \vec{f}_{\perp} \qquad \text{where} \quad \vec{\nabla}\times\vec{f}_{\parallel} = 0 \quad \text{curlfree}$$
$$\vec{\nabla}\cdot\vec{f}_{\perp} = 0 \quad \text{divergenceless}$$

where
$$\begin{cases} \vec{f}_{\parallel}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r}) = -\vec{\nabla}\int \dfrac{d^3r'}{4\pi}\,\dfrac{[\vec{\nabla}'\cdot\vec{f}(\vec{r}')]}{|\vec{r}-\vec{r}'|} \\ \vec{f}_{\perp}(\vec{r}) = \vec{\nabla}\times\vec{W}(\vec{r}) = \vec{\nabla}\times\int \dfrac{d^3r'}{4\pi}\,\dfrac{[\vec{\nabla}'\times\vec{f}(\vec{r}')]}{|\vec{r}-\vec{r}'|} \end{cases}$$

where in above we used $\quad \vec{D}(\vec{r}') = \frac{1}{4\pi}\vec{\nabla}'\cdot\vec{f}(\vec{r}')$

$$\vec{C}(\vec{r}') = \frac{1}{4\pi}\vec{\nabla}'\times\vec{f}(\vec{r}')$$

~~where~~ $\vec{f}_{\parallel}$ is called the longitudinal part of $\vec{f}$

$\vec{f}_{\perp}$ is called the transverse part of $\vec{f}$

To understand the reason for these names, we need to consider the Fourier transforms

Above can be generalized to situations where $\vec{f}$ satisfies other boundary conditions, say has a specified value on a given boundary surface. One first replaces $\dfrac{1}{|\vec{r}-\vec{r}'|}$ by the appropriate Greens function — see more to come!

Discussion regarding Fourier transforms

$$\vec{f}(\vec{r}) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, \vec{f}(\vec{k}) \qquad \text{Fourier transf}$$

$$\vec{f}(\vec{k}) = \int_{-\infty}^{\infty} d^3r\, e^{-i\vec{k}\cdot\vec{r}}\, \vec{f}(\vec{r}) \qquad \text{inverse transf}$$

Some special cases well worth remembering

① Transform of Dirac function

$$\int d^3r\, e^{-i\vec{k}\cdot\vec{r}}\, \delta(\vec{r}-\vec{r}_0) = e^{-i\vec{k}\cdot\vec{r}_0}$$

$$\Rightarrow \quad \delta(\vec{r}-\vec{r}_0) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, e^{-i\vec{k}\cdot\vec{r}_0}$$

$$\boxed{\delta(\vec{r}-\vec{r}_0) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}_0)}}$$

or letting $\vec{r} \leftrightarrow \vec{k}$ in the above

$$\delta(\vec{k}-\vec{k}_0) = \int \frac{d^3r}{(2\pi)^3}\, e^{i\vec{r}\cdot(\vec{k}-\vec{k}_0)}$$

② Transform of Coulomb potential $\frac{1}{|\vec{r}-\vec{r}'|}$

We know

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}')$$

Suppose $f(\vec{k}) \equiv \int_{-\infty}^{\infty} d^3r\, e^{-i\vec{k}\cdot\vec{r}}\, \frac{1}{|\vec{r}-\vec{r}'|}$ is the

Fourier transf of $\frac{1}{|\vec{r}-\vec{r}'|}$

Substitute
$$\begin{cases} \dfrac{1}{|\vec{r}-\vec{r}'|} = \displaystyle\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, f(\vec{k}) \\ \delta(\vec{r}-\vec{r}') = \displaystyle\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \end{cases}$$

into above Poisson equation

$$\underbrace{\nabla^2 \int \frac{d^3k}{(2\pi)^3}}\, e^{i\vec{k}\cdot\vec{r}} f(\vec{k}) = -4\pi \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

operates only on $\vec{r}$
so move inside integral

$$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = \vec{\nabla}\cdot\left(\vec{\nabla} e^{i\vec{k}\cdot\vec{r}}\right)$$

① $$\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = \sum_{i=1}^{3} \hat{x}_i \frac{\partial}{\partial x_i} e^{i\vec{k}\cdot\vec{r}} = \sum_{i=1}^{3} \hat{x}_i\, i k_i\, e^{i\vec{k}\cdot\vec{r}}$$

$$= i\vec{k}\, e^{i\vec{k}\cdot\vec{r}} \qquad \text{where } \hat{x}_1, \hat{x}_2, \hat{x}_3 = \hat{x}, \hat{y}, \hat{z}$$

② $$\vec{\nabla}\cdot\left(i\vec{k} e^{i\vec{k}\cdot\vec{r}}\right) = (i\vec{k})\cdot(i\vec{k})\, e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

so $$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

Poisson eqn then gives

$$\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} (-k^2) f(\vec{k}) = -4\pi \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, e^{-i\vec{k}\cdot\vec{r}'}$$

$$\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} \left[-k^2 f(\vec{k})\right] = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} \left[-4\pi e^{-i\vec{k}\cdot\vec{r}'}\right]$$

As is true for Fourier series, so it is true for Fourier transforms: If two functions are equal, then their Fourier transforms are equal.

$$\Rightarrow \quad -k^2 f(\vec{k}) = -4\pi e^{-i\vec{k}\cdot\vec{r}'}$$

$$\boxed{f(\vec{k}) = \frac{4\pi}{k^2} e^{-i\vec{k}\cdot\vec{r}'}}$$

$\Rightarrow$ is the Fourier transform of $\frac{1}{|\vec{r}-\vec{r}'|}$

Electrostatic

$-\nabla^2 \phi = 4\pi\rho$ with $\vec{E} = -\vec{\nabla}\phi$ (statics only)

physical meaning of the potential ϕ

work done to move a test charge δq from $\vec{r}_1$ to $\vec{r}_2$ in presence of an electric field $\vec{E}$ is

$$W_{12} = \int_{\vec{r}_1}^{\vec{r}_2} d\vec{l} \cdot \vec{F}$$

where $\vec{F}$ is the force required to move the charge. Since $\vec{E}$ exerts a force $\delta q \vec{E}$ on the charge, $\vec{F}$ must counterbalance this electric force so we can move the charge quasi statically $\Rightarrow \vec{F} = -\delta q \vec{E}$

$$W_{12} = -\delta q \int_{\vec{r}_1}^{\vec{r}_2} d\vec{l} \cdot \vec{E} = \delta q \int_{\vec{r}_1}^{\vec{r}_2} d\vec{l} \cdot \vec{\nabla}\phi = \delta q \left[\phi(\vec{r}_2) - \phi(\vec{r}_1) \right]$$

$$\phi(\vec{r}_2) - \phi(\vec{r}_1) = \frac{W_{12}}{\delta q}$$

difference in potential between two points is the work per unit charge to move a test charge between the two points

Green's Functions – part I

$$-\nabla^2\phi = 4\pi\rho$$

We already know that for a point charge q at position $\vec{r}'$, ie $\rho(\vec{r}) = q\,\delta(\vec{r}-\vec{r}')$, the solution to the above is

$$\phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{r}'|} \qquad \text{ie} \quad -\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = 4\pi\delta(\vec{r}-\vec{r}')$$

We call the special solution for a point source the Green function for the differential operator

$$-\nabla^2 G(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$$

$G(\vec{r},\vec{r}')$ gives the potential at position $\vec{r}$ due to a unit source at position $\vec{r}'$

Generally, one also has to specify a desired boundary condition for the Green function on the boundary of the system.

For the Coulomb solution for a point charge the implicit boundary condition is that the potential vanish infinitely far from the charge

$$G(\vec{r},\vec{r}') \to 0 \quad \text{as} \quad |\vec{r}-\vec{r}'| \to \infty$$

boundary of the system is taken to infinity

If one knows the Green's function, then one can find the solution for any distribution of sources $\rho(\vec{r})$

$$\phi(\vec{r}) = \int d^3r' \, G(\vec{r}, \vec{r}') \, \rho(\vec{r}')$$

<u>proof:</u>

$$-\nabla^2\phi = \int d^3r' \left[-\nabla^2 G(\vec{r}, \vec{r}')\right] \rho(\vec{r}')$$

$$= \int d^3r' \left[4\pi\,\delta(\vec{r}-\vec{r}')\right] \rho(\vec{r}')$$

$$= 4\pi\,\rho(\vec{r})$$

We will return to concept of Greens function when we discuss solution of Poisson's eqn in a finite volume

We will also see Greens functions again when we discuss solution of the inhomogeneous wave equation.