Specifying both ϕ and $\frac{\partial \phi}{\partial n}$ on surface is known as "Cauchy" boundary conditions — for Laplace's equ, Cauchy b.c. overspecify the problem + a solution cannot in general be found.

Uniqueness

If we have a system of charges in vol V, and *either* the potential ϕ, *or* its normal derivative $\frac{\partial \phi}{\partial n}$, is specified on the surfaces of V, then there is a unique solution to Poisson's equation inside V. Specifying ϕ is known as Dirichlet boundary conditions. Specifying $\frac{\partial \phi}{\partial n}$ is known as Neumann boundary conditions.

proof: Suppose we had two solutions ϕ_1 and ϕ_2, *both* with $-\nabla^2\phi = 4\pi\rho$ inside V, and obeying specified b.c. on surface of V.

Define $u = \phi_2 - \phi_1 \Rightarrow \nabla^2 u = 0$ inside V

and $u = 0$ on surface S – for Dirichlet b.c.

or $\frac{\partial u}{\partial n} = 0$ on surface S – for Neumann b.c.

Use Green's 1st identity with $\phi = \psi = u$

$$\int_V d^3r \, (\underbrace{u\nabla^2 u}_{\substack{\shortparallel \\ 0 \\ \text{as } \nabla^2 u = 0}} + \vec{\nabla}u \cdot \vec{\nabla}u) = \oint_S da \, \underbrace{u\frac{\partial u}{\partial n}}_{\substack{\shortparallel \\ 0 \\ \text{as } u \text{ or } \frac{\partial u}{\partial n} = 0}}$$

$$\Rightarrow \int_V d^3r \, |\vec{\nabla} u|^2 = 0 \quad \Rightarrow \vec{\nabla} u = 0$$

$$\Rightarrow u = \text{const}$$

For Dirichlet b.c., $u = 0$ on surface S, so const $= 0$ and $\phi_1 = \phi_2$. Solution is unique

For Neumann b.c., ϕ_1 and ϕ_2 differ only by an arbitrary constant. Since $\vec{E} = -\vec{\nabla}\phi$, the electric fields $\vec{E}_1 = -\vec{\nabla}\phi_1$ and $E_2 = -\vec{\nabla}\phi_2$ are the same.

~~Solution~~ If boundary ~~solution~~ surface S consists of several disjoint pieces, then solution is unique if specify ϕ on some pieces and $\frac{\partial \phi}{\partial n}$ on other pieces.

Solution of Poisson's equation with both ϕ and $\frac{\partial \phi}{\partial n}$ specified on the same surface S (Cauchy b.c.) does not in general exist, since specifying either ϕ or $\frac{\partial \phi}{\partial n}$ alone is enough to give a unique solution.

Green's function - part II

Greens 2nd identity

$$\int_V d^3r' \left(\phi \nabla'^2 \psi - \psi \nabla'^2 \phi \right) = \oint_S da' \left(\phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'} \right)$$

Apply above with $\begin{cases} \phi(\vec{r}') \text{ electrostatic potential with } \nabla'^2 \phi = -4\pi\rho(\vec{r}') \\ \psi(\vec{r}') = G(\vec{r},\vec{r}') \text{ the Green function satisfying} \end{cases}$

$$\nabla'^2 G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$$

we saw one solution of above is $G(\vec{r},\vec{r}') = \dfrac{1}{|\vec{r}-\vec{r}'|}$
but a more general solution is

$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} + F(\vec{r},\vec{r}')$$

where $\nabla'^2 F(\vec{r},\vec{r}') = 0$, for $\vec{r}'$ in volume V
we will choose $F(\vec{r},\vec{r}')$ to simplify solution of ϕ

$$\Rightarrow \int_V d^3r' \left(\phi(\vec{r}')\, \nabla'^2 G(\vec{r},\vec{r}') - G(\vec{r},\vec{r}') \nabla'^2 \phi(\vec{r}') \right)$$

$$= \int_V d^3r' \left(\phi(r') \left[-4\pi\delta(\vec{r}-\vec{r}')\right] - G(\vec{r},\vec{r}') \left[-4\pi\rho(\vec{r}')\right] \right)$$

$$= -4\pi\,\phi(\vec{r}) + 4\pi \int_V d^3r'\, G(\vec{r},\vec{r}')\,\rho(\vec{r}')$$

$$= \oint_S da' \left(\phi \frac{\partial G}{\partial n'} - G \frac{\partial \phi}{\partial n'} \right)$$

$$\phi(\vec{r}) = \int_V d^3r' \, G(\vec{r},\vec{r}')\,\rho(\vec{r}') + \oint_S \frac{da'}{4\pi}\left(G(\vec{r},\vec{r}')\frac{\partial\phi(\vec{r}')}{\partial n'} - \phi(r')\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\right)$$

Consider <u>Dirichlet boundary problem</u>. If we can choose $F(\vec{r},\vec{r}')$ such that $G_D(\vec{r},\vec{r}') = 0$ for $\vec{r}'$ on the boundary surface S', then above simplifies to

$$\left[\phi(\vec{r}) = \int_V d^3r' \, G_D(\vec{r},\vec{r}')\,\rho(\vec{r}') - \oint_S \frac{da'}{4\pi}\,\phi(r')\,\frac{\partial G_D(\vec{r},\vec{r}')}{\partial n'}\right]$$

Since $\rho(r)$ is specified in V, and $\phi(r)$ is specified on S, above then gives desired solution for $\phi(r)$ inside volume V.

Finding G_D is therefore equivalent to finding an $F(r,r')$ such that $\nabla'^2 F(\vec{r},\vec{r}') = 0$ for $\vec{r}'$ in V (solves Laplace eqn) and

$F(\vec{r},\vec{r}') = \dfrac{-1}{|\vec{r}-\vec{r}'|}$ for $\vec{r}'$ on boundary surface S'

Always exists unique solution for F

Next consider Neumann boundary problem.

One might think to find $F(\vec{r},\vec{r}\,')$ such that $\dfrac{\partial G(\vec{r},\vec{r}\,')}{\partial n'} = 0$ on boundary surface. But this is not possible.

Consider $\displaystyle\int_V \nabla'^2 G(\vec{r},\vec{r}\,')\, d^3r' = \int \vec{\nabla}'\cdot\vec{\nabla}' G(\vec{r},\vec{r}\,')\, d^3r'$

$$= \oint_S \vec{\nabla}' G(\vec{r},\vec{r}\,')\cdot\hat{n}\, da'$$

$$= \oint_S \frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'}\, da' = -4\pi$$

since $\nabla'^2 G = -4\pi\delta(r-r')$

So we cant have $\dfrac{\partial G}{\partial n'} = 0$ for $\vec{r}\,'$ on S

Simplest choice is then $\dfrac{\partial G_N(r,r')}{\partial n'} = -\dfrac{4\pi}{S}$ for $\vec{r}\,'$ on S

area of surface

Then

$$\phi(\vec{r}) = \int_V d^3r'\, G_N(\vec{r},\vec{r}\,')\,\rho(\vec{r}\,') + \oint \frac{da'}{4\pi}\, G(\vec{r},\vec{r}\,')\,\frac{\partial\phi(\vec{r}\,')}{\partial n'}$$

$$+ \oint \frac{da'}{4\pi}\,\phi(r')\left(\frac{-4\pi}{S}\right)$$

$$\left[\phi(\vec{r}\,') = \int_V d^3r'\, G_N(\vec{r},\vec{r}\,')\,\rho(\vec{r}\,') + \oint \frac{da'}{4\pi}\, G_N(\vec{r},\vec{r}\,')\,\frac{\partial\phi(\vec{r}\,')}{\partial n'}\right]$$

$$+ \langle\phi\rangle_S$$

Since $\rho(r)$ is specified in V
and $\dfrac{\partial\phi}{\partial n}$ is specified on S

constant = average value of ϕ on surface S.

above gives solution $\phi(r)$ in V within additive constant $\langle\phi\rangle_S$
Since $\vec{E} = -\vec{\nabla}\phi$, the const $\langle\phi\rangle_S$ is of no consequence

Finding $G_N(\vec{r},\vec{r}')$ is therefore equivalent to finding an $F(\vec{r},\vec{r}')$ such that

$$\nabla'^2 F(\vec{r},\vec{r}') = 0 \quad \text{for } \vec{r}' \text{ in } V$$

$$\text{and} \quad \frac{\partial F(\vec{r},\vec{r}')}{\partial n'} = \frac{-4\pi}{S'} \quad \text{for } \vec{r}' \text{ on surface } S'$$

always exists a unique solution (within additive constant)

While G_D and G_N always exist in principle, they depend in detail on the shape of the surface S' and are difficult to find except for simple geometries

In preceeding we defined G by $\nabla'^2 G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}')$

But our earlier interpretation of $G(\vec{r},\vec{r}')$ was that it was potential at $\vec{r}$ due to point source at $\vec{r}'$, ie $\nabla^2 G(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$. Note, for general surface S', $G(\vec{r},\vec{r}')$ is not in general a function of $|\vec{r}-\vec{r}'|$ but depends on $\vec{r}$ and $\vec{r}'$ separately. But the equivalence of the two definitions of G above is obtained by noting that one can prove the symmetry property

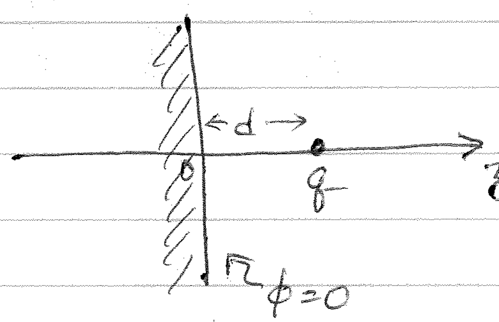
$$G(\vec{r},\vec{r}') = G(\vec{r}',\vec{r})$$

for Dirichlet b.c., and one can impose it as an additional requirement for Neumann b.c. (see Jackson, end section 1.10)

Image charge method

For simple geometries, can try to obtain G_D or G_N by placing a set of "image charges" outside the volume of interest V, i.e. on the "other side" of the system boundary surface S. Because these image charges are outside V, their contrib to the potential inside V obeys $\nabla^2\phi^{image} = 0$, as necessary. Choose location of image charges so that total ϕ has desired boundary condition.

1) charge in front of infinite grounded plane



want $\nabla^2\phi = -4\pi q\,\delta(x)\delta(y)\delta(z-d)$

$\phi = 0$ for $z = 0$

If we find a solution to above it is the unique solution

Solution - put fictitious image charge $-q$ at $z = -d$

ϕ is Coulomb potential from the real charge + the image

$$\phi(\vec{r}) = \underbrace{\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}}_{\text{real charge}} + \underbrace{\frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}}_{\text{image charge}}$$

above satisfies $\phi(x,y,0) = 0$ as required

Also, $\nabla^2\phi = -4\pi q\,\delta(\vec{r}-d\hat{z}) + 4\pi q\,\delta(\vec{r}+d\hat{z})$

$= -4\pi q\,\delta(\vec{r}-d\hat{z})$ for region $z > 0$

Can now find $\vec{E}$ for $z > 0$

$$\vec{E} = -\vec{\nabla}\phi$$

In particular $E_z = -\frac{\partial \phi}{\partial z} = +q\left[\left(\frac{1}{2}\right)\frac{2(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \left(\frac{1}{2}\right)\frac{2(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$

$$E_z = q\left[\frac{(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

We can use above to compute the surface charge density $\sigma(x,y)$ induced on the surface of the conducting plane. At conductor surface

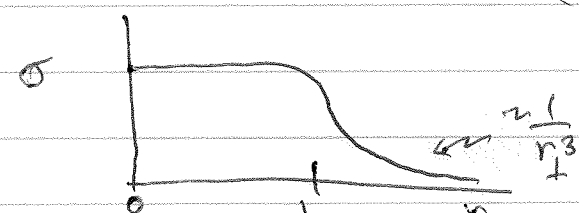
$$-\frac{\partial \phi}{\partial n} = 4\pi\sigma$$

$$\Rightarrow \quad \sigma = -\frac{1}{4\pi}\frac{\partial \phi}{\partial z} = \frac{1}{4\pi}E_z(x,y,z=0)$$

$$\sigma(x,y) = \frac{q}{4\pi}\left[\frac{-d}{(x^2+y^2+d^2)^{3/2}} - \frac{d}{(x^2+y^2+d^2)^{3/2}}\right]$$

$$= -\frac{q}{2\pi}\frac{d}{(x^2+y^2+d^2)^{3/2}} = \frac{-qd}{2\pi\,(r_\perp^2+d^2)^{3/2}}$$

$$r_\perp = \sqrt{x^2+y^2}$$



Total induced charge is

$$q^{induced} = \int_{-\infty}^{\infty} dx\,dy\ \sigma(x,y)$$

$$= 2\pi \int_0^{\infty} dr_\perp\, r_\perp\ \sigma(r_\perp)$$

$$= 2\pi \int_0^{\infty} dr_\perp\ \frac{r_\perp\ (-qd)}{2\pi\,(r_\perp^2+d^2)^{3/2}}$$

$$= -qd\left[\frac{-1}{(r_\perp^2+d^2)^{1/2}}\right]_0^{\infty}$$

$$= -qd\left[0 - \frac{-1}{d}\right]$$

$$q^{induced} = -q$$ induced charge = image charge

Force on charge q in front of conducting plane is due to the induced σ. The E field of this σ is, for $z>0$, the same as the E field of the image charge.

$$\Rightarrow \vec{F} = \frac{-q^2}{(2d)^2}\,\hat{z} = \frac{-q^2}{4d^2}\,\hat{z}$$ <u>attractive</u>

Work done to move q into position from infinity is

$$W = -\int_{\infty}^{d} d\vec{l}\cdot\vec{F} = -\int_{\infty}^{d} dz\ F_z$$

we must oppose electrostatic force $\vec{F}$

$$W = \int_d^\infty dz \left(\frac{-q^2}{4z^2} \right) = \frac{-q^2}{4d}$$

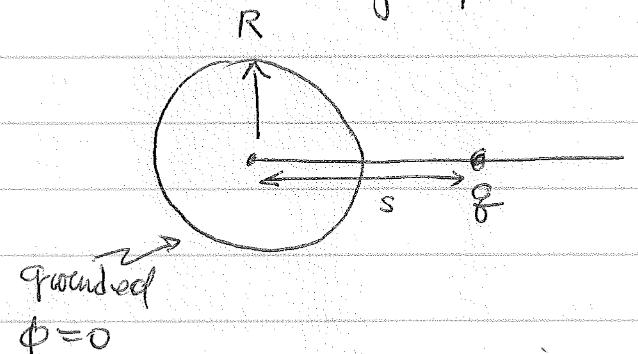
$W < 0 \Rightarrow$ energy released

Note: W above is not the electrostatic energy that would be present if the image charge were real

i.e. it is not $q\phi^{\text{image}}(\vec{r} = d\hat{z}) = \frac{-q^2}{2d}$

One way to see why is to note that as q is moved quasistatically in towards the conducting plane, the image charge also must be moving to stay equidistant on the opposite side,

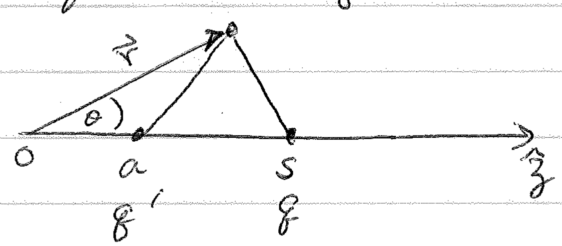
2) point charge in front of a grounded ($\phi = 0$) conducting sphere.



charge q placed a distance s from center of grounded conducting sphere of radius R

place image charge q' inside sphere so that the combined ϕ from q and q' vanishes on surface of sphere.

By symmetry, q' should lie on the same radial line as q does. Call the distance of q' from the origin "a"



potential at position $\vec{r}$ is

$$\phi(\vec{r}) = \frac{q}{|\vec{r} - s\hat{z}|} + \frac{q'}{|\vec{r} - a\hat{z}|}$$

$$= \frac{q}{(r^2 + s^2 - 2sr\cos\theta)^{1/2}} + \frac{q'}{(r^2 + a^2 - 2ra\cos\theta)^{1/2}}$$

Can we choose q' and a so that $\phi(R, \theta) = 0$ for all θ?

$$\phi(R,\theta) = \frac{q}{(R^2+s^2-2sR\cos\theta)^{1/2}} + \frac{q'}{(R^2+a^2-2aR\cos\theta)^{1/2}}$$

make denominators look alike

$$R^2+a^2-2aR\cos\theta = \frac{a}{s}\left(\frac{s}{a}R^2 + sa - 2sR\cos\theta\right)$$

if choose $sa = R^2$, ie $\boxed{a = R^2/s}$, then $\frac{sR^2}{a} = s^2$
and then the denominator of the 2nd term is

$$\left[\frac{R^2}{s^2}(s^2+R^2-2sR\cos\theta)\right]^{1/2} = \frac{R}{s}\left[s^2+R^2-2sR\cos\theta\right]^{1/2}$$

$$\Rightarrow \quad \phi(R,\theta) = \frac{q}{(R^2+s^2-2sR\cos\theta)^{1/2}} + \frac{q'(s/R)}{(R^2+s^2-2sR\cos\theta)^{1/2}}$$

So choose $q'(s/R) = -q \rightarrow \boxed{q' = -qR/s}$
to get $\phi(R,\theta)=0$.

Solution is

$$\phi(r,\theta) = \frac{q}{(r^2+s^2-2rs\cos\theta)^{1/2}} - \frac{qR/s}{\left(r^2+\frac{R^4}{s^2}-2r\frac{R^2}{s}\cos\theta\right)^{1/2}}$$

$$= \frac{q}{(r^2+s^2-2rs\cos\theta)^{1/2}} - \frac{q}{\left(\frac{s^2r^2}{R^2}+R^2-2rs\cos\theta\right)^{1/2}}$$

Can get induced surface charge on sphere by

$$4\pi\sigma = \vec{E}\cdot\hat{r} = -\left.\frac{\partial\phi}{\partial r}\right|_{r=R}$$ see Jackson Eq (2.5) for result

$$\sigma(\theta) = -\frac{q}{4\pi}\frac{1}{RS}\frac{1-(R/s)^2}{\left(1+(R/s)^2-2(R/s)\cos\theta\right)^{3/2}}$$

$\sigma(\theta)$ is greatest at $\theta=0$, as one should expect

Can integrate $\sigma(\theta)$ to get total induced charge. One finds

$$2\pi\int_0^\pi d\theta \sin\theta\, R^2\, \sigma(\theta) = q' = -qR/s$$

In general, total induced charge = sum of all image charges

Force of attraction of charge to sphere

Force on q is due to electric field from induced charge σ which is the same as the electric field from the image charge q'.

$$\vec{F} = \frac{qq'\,\hat{z}}{(s-a)^2} = \frac{-q^2(R/s)\,\hat{z}}{(s-R^2/s)^2} = \frac{-q^2Rs}{(s^2-R^2)^2}\,\hat{z}$$

Close to the surface of the sphere, $s\approx R$, so write $s=R+d$ where $d\ll R$. Then

$$\vec{F} = \frac{-q^2Rs}{(s-R)^2(s+R)^2} = \frac{-q^2R(R+d)}{d^2\,(2R+d)^2} \simeq \frac{-q^2}{4d^2}$$

get same result as for infinite flat grounded plane.
When q is so close to surface that $d\ll R$, the charge does not "see" the curvature of the surface