far from the surface, $s \gg R$

$$\vec{F} = \frac{q q' \hat{z}}{(s-a)^2} = \frac{-q^2 R s}{(s^2 - R^2)^2} \hat{z} \simeq -\frac{q^2 R}{s^3} \hat{z}$$

$F \sim \frac{1}{s^3}$ very different from flat plane
also different from point charge

<u>Note</u>: In preceding two problems, what we found was a ϕ such that $\nabla^2 \left(\frac{\phi}{q}\right) = -4\pi \delta(\vec{r} - \vec{r}_0)$, for a charge at $\vec{r}_0$, and $\phi = 0$ on the boundary. Such a ϕ is nothing more than G_D the corresponding Green function for Dirichlet boundary conditions.

Suppose now that instead of a grounded sphere we have a sphere with fixed net charge Q.

We want to add new image charge to represent this case. If we put $q' = -q R/s$ at $a = R/s$ as before, the boundary condition of $\phi = \text{const}$ on surface $r = R$ is met, but the net charge on the sphere is q' (the induced charge) not the desired Q. We therefore need to add new image charge(s) of total charge $Q - q'$ (so total image charge is Q) in such a way that we keep ϕ constant on the surface of the sphere. The way to do this is to put $Q - q'$ at the origin!

Solution is

$$\phi(r,\theta) = \frac{Q + qR/s}{r} + \frac{q}{(r^2+s^2-2rs\cos\theta)^{1/2}} - \frac{q}{\left(\frac{s^2r^2}{R^2}+R^2-2rs\cos\theta\right)^{1/2}}$$

The force on the charge q is due to the $\vec{E}$ field of the images

$$\vec{F} = F\hat{z} = \frac{q(Q+qR/s)\hat{z}}{s^2} + \frac{qq'\,\hat{z}}{(s-a)^2}$$

$$F = \frac{qQ}{s^2} + \frac{q^2R/s}{s^2} - \frac{q^2R/s}{(s-R^2/s)^2}$$

$$= \frac{qQ}{s^2} + q^2R\left[\frac{1}{s^3} - \frac{1}{s^3(1-R^2/s^2)^2}\right]$$

$$= \frac{qQ}{s^2} + \frac{q^2R}{s^3}\left[1 - \frac{1}{(1-R^2/s^2)^2}\right]$$

$$F = \frac{qQ}{s^2} - q^2\frac{R^3}{s}\,\frac{2-R^2/s^2}{(s^2-R^2)^2}$$

$$F = \frac{qQ}{s^2} - \frac{q^2R^3}{s}\frac{(2-R^2/s^2)}{(s^2-R^2)^2}$$ sphere with fixed charge Q

special case $Q=0$, a <u>neutral</u> conducting sphere

then $$F = -\frac{q^2R^3}{s}\frac{(2-R^2/s^2)}{(s^2-R^2)^2}$$

For large $s \gg R$, far from surface, we have

$$F = -\frac{2q^2R^3}{s^5} \sim \frac{1}{s^5} \quad \text{attractive}$$

compare to force from the <u>grounded</u> sphere

$$F = -\frac{q^2R}{s^3} \sim \frac{1}{s^3}$$

<u>we see there is a very big difference between a grounded and a neutral sphere!</u>

Return to case with general charge Q

For large $s \gg R$ far from surface

$$F \simeq \frac{qQ}{s^2} - \frac{2q^2R^3}{s^5}$$

The leading term is just the Coulomb force between q and Q at the origin

For $Q>0$, F is always positive, ie <u>repulsive</u>, for large enough s.

For $s = R+d$, $d \ll R$ close to surface

$$F = \frac{qQ}{(R+d)^2} - \frac{q^2 R^3}{(R+d)} \frac{2 - \frac{R^2}{(R+d)^2}}{(R^2+d^2+2Rd-R^2)^2}$$

$$\simeq \frac{qQ}{R^2} - \frac{q^2 R^3}{R} \frac{(2-1)}{4R^2 d^2}$$

$$F \simeq \frac{qQ}{R^2} - \frac{q^2}{4d^2} \simeq -\frac{q^2}{4d^2} \quad \text{for } d \text{ small enough}$$

F is always attractive for small enough d, and is equal to the force in front of a grounded plane, no matter what is the value of Q! This is because the image charge q' lies so much closer to q than does the $Q-q'$ at the origin, that it dominates the force.

The cross over from attractive to repulsive occurs at a distance s that depends on Q. This distance is given by

$$\frac{Q}{q} = \frac{R^3 s \,(2 - R^2/s^2)}{(s^2 - R^2)^2} = \left(\frac{R}{s}\right)^3 \frac{2 - (R/s)^2}{\left[1 - (R/s)^2\right]^2}$$

let $x = R/s \in (0,1)$

$$\frac{Q}{q} = x^3 \frac{(2 - x^2)}{(1-x^2)^2}$$

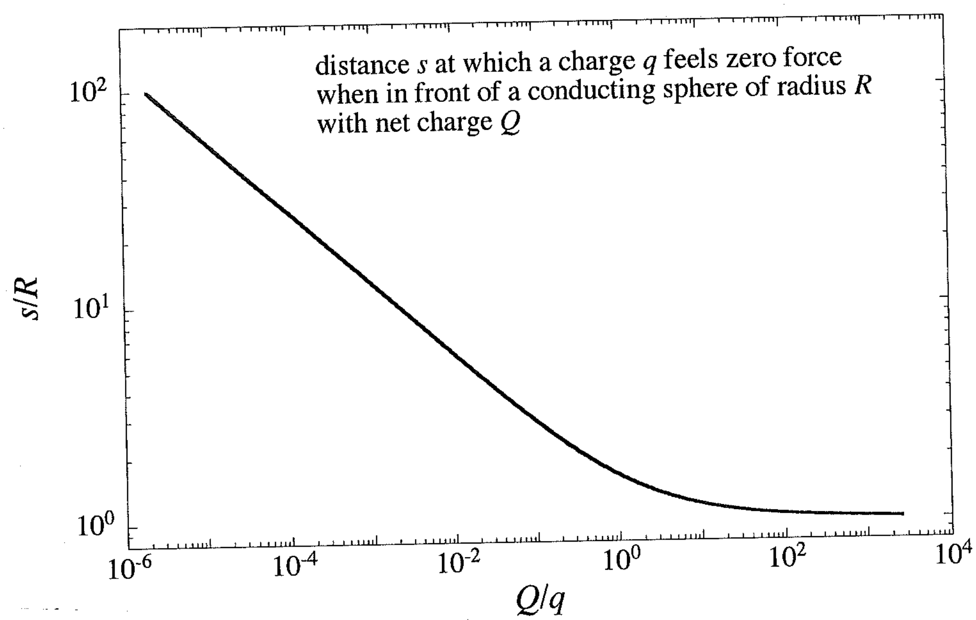
gives 5th order polynomial in x
no analytic solution
can solve graphically

For $\frac{Q}{q} = 1$, cross over is at $\frac{R}{s} = 0.62$

$$s = 1.6 R$$

$\frac{Q}{q} = 0.1$ cross over is at $\frac{R}{s} \simeq 0.36$

$$s = 2.8 R$$



For q close to the surface we had $F \simeq \frac{qQ}{R^2} - \frac{q^2}{4d^2} \approx \frac{-q^2}{4d^2}$

when we try to ionize an electron from a neutral metal (this is the photo electric effect!) it is not the charge left behind that is the dominant force one has to work against, rather it is the force of attraction due to the induced surface charge, ie the $\frac{-q^2}{4d^2}$ term, that is independent of Q

Separation of Variables

If the system has a rectangular $\underbrace{\text{boundary}}_{}$ and contains $\underline{\underline{\text{no}}}$ charge, we can look for solutions to $\nabla^2\phi = 0$ of the form

$$\phi(\vec{r}) = X(x)\,Y(y)\,Z(z)$$

product of three function each of one variable only

$$\nabla^2\phi = 0 \Rightarrow \frac{1}{\phi}\nabla^2\phi = 0$$

$$\Rightarrow \frac{1}{X(x)}\frac{d^2X}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z}{dz^2} = 0$$

The only way this can happen for $\underline{\text{all}}$ values of x, y, z is if each of the three terms is a constant; call them a^2, b^2, c^2

$$\frac{1}{X}\frac{d^2X}{dx^2} = a^2 \Rightarrow X(x) = A_1e^{-ax} + A_2e^{ax}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = b^2 \qquad Y(y) = B_1e^{-by} + B_2e^{by}$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = c^2 \qquad Z(z) = C_1e^{-cz} + C_2e^{cz}$$

with $a^2 + b^2 + c^2 = 0$

$\Rightarrow$ at least one of the a^2, b^2, c^2 is <0

$\Rightarrow$ at least one of the a, b, c is $\underline{\text{imaginary}}$.

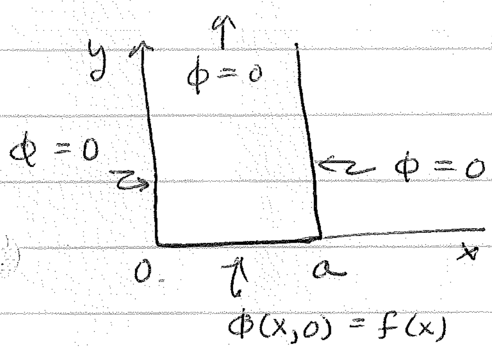
Above is one particular solution. But there are many solutions, each with different a, b, c, but all obeying the constraint $a^2+b^2+c^2=0$. The General solution is a superposition of these

$$\phi(x,y,z) = \sum_i \left(A_{1i}\, e^{-a_i x} + A_{2i}\, e^{a_i x}\right)\left(B_{1i}\, e^{-b_i y} + B_{2i}\, e^{b_i y}\right)\left(C_{1i}\, e^{-c_i z} + C_{2i}\, e^{c_i z}\right)$$

$$a_i^2 + b_i^2 + c_i^2 = 0$$

<u>Example</u>

Consider a channel shaped as below - infinite along z



$\phi(0,y) = 0$

$\phi(a,y) = 0$

$\phi(x,y) = 0$ as $y \to \infty$

$\phi(x,0) = f(x)$ specified function

solution is independent of z $\Rightarrow$

$$\phi(x,y) = \sum_i \left(A_{1i}\, e^{-a_i x} + A_{2i}\, e^{a_i x}\right)\left(B_{1i}\, e^{-b_i y} + B_{2i}\, e^{b_i y}\right)$$

$$a_i^2 + b_i^2 = 0$$

we will see that the correct thing is to choose a imaginary

let $a_i = i\alpha_i$

$b_i = \alpha_i$

$$\phi(x,y) = \sum_i \left(A_i \cos\alpha_i x + B_i \sin\alpha_i x\right)\left(C_i\, e^{-\alpha_i y} + D_i\, e^{\alpha_i y}\right)$$

where $A_i = (A_{1i} + A_{2i})$ $\qquad C_i = B_{1i}$

$B_i = i(A_{1i} - A_{2i})$ $\qquad D_i = B_{2i}$

Now $\phi(x,y) \to 0$ as $y \to \infty$ for all x $\Rightarrow$ $\boxed{D_i = 0}$

$$\Rightarrow \quad \phi(x,y) = \sum_i \left[A_i' \cos \alpha_i x + B_i' \sin \alpha_i x\right] e^{-\alpha_i y}$$

where $A_i' = A_i C_i$, $B_i' = B_i C_i$

$$\phi(0,y) = 0 \Rightarrow \sum_i A_i' e^{-\alpha_i y} = 0 \text{ all } y \Rightarrow \boxed{A_i' = 0}$$

$$\Rightarrow \quad \phi(x,y) = \sum_i B_i' \sin(\alpha_i x) e^{-\alpha_i y}$$

$$\phi(a,y) = 0 \Rightarrow \sum_i B_i' \sin(\alpha_i a) e^{-\alpha_i y} = 0 \text{ all } y$$

$$\Rightarrow \sin(\alpha_i a) = 0 \quad \text{or} \quad \alpha_i a = n\pi$$

$$\alpha_i = \frac{n\pi}{a} \quad \text{integer } n \geq 1$$

$$\Rightarrow \quad \boxed{\phi(x,y) = \sum_{n=1}^{\infty} B_n' \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}}$$

Finally

$$\phi(x,0) = f(x) \Rightarrow \underbrace{\sum_{n=1}^{\infty} B_n' \sin\left(\frac{n\pi x}{a}\right) = f(x)}_{\text{This is just the Fourier series for } f(x)!}$$

$$\boxed{B_n' = \frac{2}{a}\int_0^a dx\, f(x) \sin\left(\frac{n\pi x}{a}\right)}$$

We have thus determined all unknown coefficients and found the solution!

See Jackson 2-8 if Fourier series needs review

Recall orthogonality: $\frac{2}{a}\int_0^a dx \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$

For $f(x) = \Phi_0$ a constant,

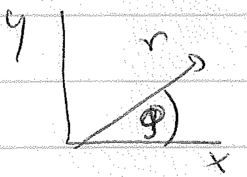
$$B_n' = \frac{2}{a}\Phi_0 \int_0^a dx \sin\left(\frac{n\pi x}{a}\right) = \frac{2\Phi_0}{a}\left[-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right]_0^a$$

$$= \frac{2\Phi_0}{n\pi}\left(1-\cos n\pi\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4\Phi_0}{n\pi} & n \text{ odd} \end{cases}$$

Polar Coordinates

– still translationally invariant along z – so two dimensional

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2} = 0$$



assume $\phi(r,\varphi) = R(r)\,\Phi(\varphi)$

$$\frac{r^2 \nabla^2 \phi}{\phi} = \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = 0$$

each term must be a constant

$$\Rightarrow \quad \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \nu^2, \quad \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -\nu^2$$

Solutions are
$$\left.\begin{aligned} R(r) &= a r^{\nu} + b r^{-\nu} \\ \Phi(\varphi) &= A\cos(\nu\varphi) + B\sin(\nu\varphi) \end{aligned}\right\} \quad \nu \neq 0$$

$$\left.\begin{aligned} R(r) &= a_0 + b_0 \ln r \\ \Phi(\varphi) &= A_0 + B_0\varphi \end{aligned}\right\} \quad \nu = 0$$

$\cancel{\Phi(r,\varphi) = (a_0 + b_0 \ln r)(A_0 + B_0\varphi) + }$

If φ can take its entire range from 0 to 2π ~~then~~ (such as problem in which ϕ is specified on the surface of a cylinder) then periodicity in $\varphi \to \varphi + 2\pi$ requires $B_0 = 0$ and $\nu =$ integer n

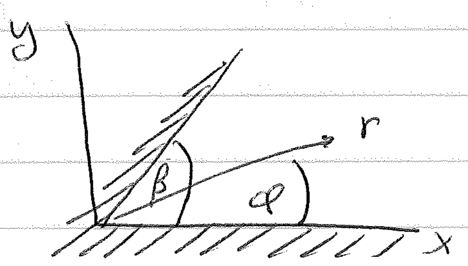
$$\phi = a_0 + b_0 \ln r + \sum_{n=1}^{\infty}\Big[r^n\left(A_n\cos(n\varphi) + B_n\sin(n\varphi)\right) + r^{-n}\left(C_n\cos(n\varphi) + D_n\sin(n\varphi)\right)\Big]$$

or reparameterizing

$$\phi(r,\varphi) = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} \left[a_n r^n \sin(n\varphi + \alpha_n) + b_n r^{-n} \sin(n\varphi + \beta_n) \right]$$

If the region where there is no charge includes $r=0$, then all $b_n = 0$ since ϕ should not diverge at the origin. If $r=0$ is excluded from the region, then the b_n need not be zero. The case $b_0 \neq 0$ corresponds to a line charge λ along the z axis.

Consider the case where φ has a restricted range, for example a wedge shaped opening of angle β



shaded region is conductor

$0 \leq \varphi \leq \beta$

ϕ is constant in conductor
$\Rightarrow$ boundary conditions

$$\begin{cases} \phi(r,0) = \phi_0 \\ \phi(r,\beta) = \phi_0 \end{cases}$$

The general solution is the linear combination

$$\phi(r,\varphi) = (a_0 + b_0 \ln r)(A_0 + B_0 \varphi)$$
$$+ \sum_{\nu > 0} \left(a_\nu r^\nu + b_\nu r^{-\nu} \right)\left(A_\nu \cos(\nu\varphi) + B_\nu \sin(\nu\varphi) \right)$$