① The condition $\phi(r,0) = \phi_0$ a constant <u>indep of r</u> then requires

$$b_0 = 0 \;, \quad A_\nu = 0 \quad \text{all } \nu$$

So

$$\phi(r,\varphi) = a_0(A_0 + B_0\varphi) + \sum_{\nu>0} (a_\nu r^\nu + b_\nu r^{-\nu}) B_\nu \sin(\nu\varphi)$$

② Since ϕ should be continuous as one approaches the conducting surface, and $\phi = \phi_0$ is a finite constant on the conducting surface, then ϕ cannot diverge as one approaches the origin $r=0$ along any fixed angle φ. This requires

$$b_\nu = 0 \quad \text{all } \nu$$

So

$$\phi(r,\varphi) = a_0(A_0 + B_0\varphi) + \sum_{\nu>0} a_\nu r^\nu \sin(\nu\varphi)$$

③ The condition $\phi(r,\beta) = \phi_0$ a constant <u>independent of r</u> then requires

$$\sin(\nu\beta) = 0 \Rightarrow \nu = \frac{n\pi}{\beta} \;, \quad n \text{ integer} \geq 1$$

So

$$\phi(r,\varphi) = a_0(A_0 + B_0\varphi) + \sum_{n=1}^{\infty} a_n r^{\frac{n\pi}{\beta}} \sin\left(\frac{n\pi\varphi}{\beta}\right)$$

④ as ϕ must approach the constant ϕ_0 as $r \to 0$ along any fixed angle φ, we therefore must have

$$B_0 = 0 \;, \quad a_0 A_0 = \phi_0$$

So finally we have

$$\phi(r,\varphi) = \phi_0 + \sum_{n=1}^{\infty} a_n r^{\frac{n\pi}{\beta}} \sin\left(\frac{n\pi\varphi}{\beta}\right)$$

We still have all the unknown a_n! These depend on how $\phi(r,\varphi)$ behaves as $r\to\infty$ (we can't make the choice here that $\phi\to 0$ as $r\to\infty$) - this is additional information that must be specified to find the complete solution.

Nevertheless we can still get very interesting information near the origin at small r. In this case, the leading term in the above series expansion for ϕ is the $n=1$ term, as it vanishes most slowly as $r\to 0$.

$$\phi(r,\varphi) \simeq \phi_0 + a_1 r^{\pi/\beta} \sin\left(\frac{\pi\varphi}{\beta}\right)$$

The electric field is

$$E_r(r,\varphi) = -\frac{\partial\phi}{\partial r} = \frac{-\pi a_1}{\beta} r^{\frac{\pi}{\beta}-1} \sin\left(\frac{\pi\varphi}{\beta}\right)$$

$$E_\varphi(r,\varphi) = -\frac{1}{r}\frac{\partial\phi}{\partial\varphi} = \frac{-\pi a_1}{\beta} r^{\frac{\pi}{\beta}-1} \cos\left(\frac{\pi\varphi}{\beta}\right)$$

$$\Rightarrow \boxed{E \sim r^{\frac{\pi}{\beta}-1}}$$

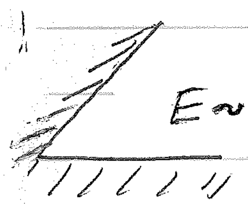
Induced surface charge given by $4\pi\sigma = \vec{E}\cdot\hat{n}$

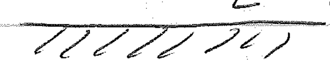
for surface at $\varphi = 0$, $\hat{n} = \hat{\varphi}$
for surface at $\varphi = \beta$, $\hat{n} = -\hat{\varphi}$

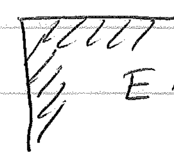
$$\sigma(r, \varphi = 0) = \frac{E_\varphi(r,0)}{4\pi} = -\frac{a_1}{4\beta} r^{\frac{\pi}{\beta}-1}$$

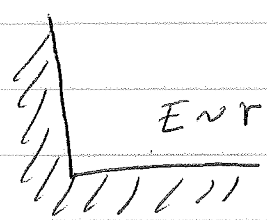
$$\sigma(r, \varphi = \beta) = \frac{-E_\varphi(r,\beta)}{4\pi} = -\frac{a_1}{4\beta} r^{\frac{\pi}{\beta}-1}$$

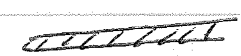
For $\frac{\pi}{\beta} > 1$, ie $\beta < \pi$, $\vec{E}$ and σ vanish as approach the origin.
For $\frac{\pi}{\beta} < 1$, ie $\beta > \pi$, $\vec{E}$ and σ diverge as approach the origin

$\beta = \frac{\pi}{4}$  $E \sim r^3$

$\beta = \pi$  $\vec{E} \sim$ const

$\beta = \frac{3\pi}{2}$  $E \sim r^{-1/3}$

$\beta = \frac{\pi}{2}$  $E \sim r$

$\beta = 2\pi$  $E \sim r^{-1/2}$

E diverges at an "extrnal" corner
E vanishes at an "internal" corner

Remember, the above examples all had translational symmetry along z, so the "corners" above are really infinitely long straight "edges".

Spherical Coordinates

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0$$

$$\phi(r,\theta,\varphi) = R(r)\,\Theta(\theta)\,\Phi(\varphi)$$

$$r^2\nabla^2\phi = \Theta\Phi\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{R\Theta}{\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = 0$$

$$\frac{r^2\sin^2\theta\,\nabla^2\phi}{\phi} = \underbrace{\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{\substack{\text{depends only on } r \text{ and } \theta \\ = -\text{const}}} + \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}_{\substack{\text{depends only on } \varphi \\ = \text{const}}} = 0$$

take $\displaystyle\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -m^2$

$$\Rightarrow \boxed{\Phi = e^{\pm im\varphi}}$$

m integer for 2π periodicity in φ

$$\Rightarrow \frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2$$

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\substack{\text{depends only on } r \\ = \text{const}}} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}}_{\substack{\text{depends only on } \theta \\ = -\text{const}}} = 0$$

call the const $\ell(\ell+1)$

For R

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \ell(\ell+1) = 0$$

Solutions are of the form $\boxed{R(r) = a_\ell r^\ell + b_\ell r^{-(\ell+1)}}$

substitute in to verify

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{d}{dr}\left(r^2\left(\ell a_\ell r^{\ell-1} - (\ell+1)b_\ell r^{-\ell-2}\right)\right)$$

$$= \frac{d}{dr}\left(\ell a_\ell r^{\ell+1} - (\ell+1)b_\ell r^{-\ell}\right)$$

$$= \ell(\ell+1)a_\ell r^\ell + \ell(\ell+1)b_\ell r^{-(\ell+1)} = \ell(\ell+1)R$$

For Θ :

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -\ell(\ell+1)$$

let $x = \cos\theta$

$dx = -\sin\theta\, d\theta$

$d\theta = -\dfrac{dx}{\sin\theta}$

$0 \leq \theta \leq \pi$

solutions for $-1 \leq x \leq 1$
correspond to $\ell \geq 0$ integers

above becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]\Theta = 0$$

Called generalized Legendre Equation - solutions are called the associated Legendre functions.

ordinary Legendre polynomials are solutions for $m=0$

For the special case $m=0$, ie the solution has azimuthal symmetry and Φ does not depend on the angle φ (ie rotational symmetry about $\hat{z}$ axis),

We want the solutions to

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \ell(\ell+1)\Theta = 0$$

The solutions are known as the Legendre polynomials, $P_\ell(x)$.

They are given, for ℓ integer, by

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell \quad \text{Rodriguez's formula}$$

The lowest ℓ polynomials are

$$P_0(x) = 1 \qquad P_2(x) = \tfrac{1}{2}(3x^2-1)$$
$$P_1(x) = x \qquad P_3(x) = \tfrac{1}{2}(5x^3-3x)$$

In general, $P_\ell(x)$ is a polynomial of order ℓ with only even powers if ℓ is even, and only odd powers if ℓ is odd. $\Rightarrow P_\ell(x) \begin{cases} \text{even in } x \text{ for } \ell \text{ even} \\ \text{odd in } x \text{ for } \ell \text{ odd} \end{cases}$

$P_\ell(x)$ is normalized so that $P_\ell(1) = 1$

Note: Legendre polynomials are only for integer $\ell \geq 0$.
What about solutions for non integer ℓ?
The $P_\ell(x)$ give one solution for each integer ℓ.
But $P_\ell(x)$ are defined by a 2nd order differential equation – shouldn't there be a 2nd independent solution for each ℓ?
It turns out that these "2nd" solutions, as well as solutions for non integer ℓ, all blow up at either $x=-1$ or $x=1$, ie at $\theta=0$ or $\theta=\pi$.
They therefore are physically unacceptable and we do not need to consider them. See Jackson 3.2

The Legendre polynomials are orthogonal and form a complete set of basis functions on the interval $-1 \leq x \leq 1$.

$$\int_{-1}^{1} dx\, P_\ell(x) P_m(x) = \int_0^\pi d\theta \sin\theta\, P_\ell(\cos\theta) P_m(\cos\theta) = \begin{cases} 0 & \ell \neq m \\ \dfrac{2}{2\ell+1} & \ell = m \end{cases}$$

$\Rightarrow$ we can expand any function $f(\theta)$, $0 \leq \theta \leq \pi$, as a linear combination of the $P_\ell(\cos\theta)$.
This is the reason they are useful for solving problems of Laplace's eqn with spherical boundary surfaces.

For $m \neq 0$, the solutions to (see Jackson 3.5)

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]\Theta = 0$$

are the associated Legendre functions $P_\ell^m(x)$.
For $P_\ell^m(x)$ to be finite in interval $-1 \leq x \leq 1$
one again finds that ℓ must be integer $\ell \geq 0$, and integer m must satisfy $|m| \leq \ell$, ie $m = -\ell, -\ell+1, \ldots, 0, \ldots \ell-1, \ell$.
For each ℓ and m there is only one such non divergent solution.

It is typical to combine the solutions $P_\ell^m(\cos\theta)$ to the θ-part of the equation with the $\Phi_m(\varphi) = e^{im\varphi}$ solutions to the φ-part of the equation to define the spherical harmonics

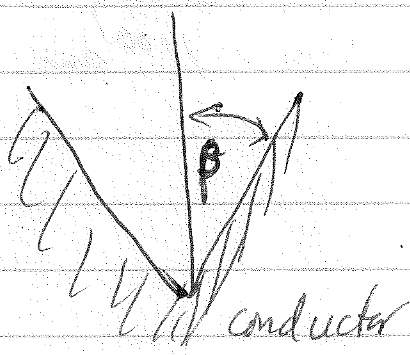
$$Y_{\ell m}(\theta, \varphi) \equiv \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\varphi}$$

The $Y_{\ell m}$ are orthogonal

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \; Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}$$

and are a complete set of basis functions for expanding any function $f(\theta,\varphi)$ defined on the surface of a sphere.

Behavior of fields near conical hole or sharp tip



we now want to solve the $\nabla^2\phi = 0$ with separation of variables, but now θ is restricted to range $0 \le \theta \le \beta$.

We still have azimuthal symmetry, but now, since we do not need solution to be finite for all $\theta \in [0, \pi]$, but only $\theta \in [0, \beta]$, we have more solutions to the ⓗ equation, ie l does not have to be integer, - still need $l \geq 0$ to be finite at $\theta = 0$.

see Jackson sec. 3.4 for details.