

i th component of integrand on right hand side is (\vec{E} part only)
(sum over repeated indicies)

$$\begin{aligned}
 & E_i \partial_j E_j - \epsilon_{ijkl} E_j E_{klm} \partial_l E_m \\
 &= E_i \partial_j E_j - (\delta_{cl} \delta_{jm} - \delta_{cm} \delta_{jl}) E_j \partial_l E_m \\
 &= E_i \partial_j E_j - E_j \partial_i E_j + E_j \partial_j E_i \\
 &= \partial_j (E_i E_j - \frac{1}{2} \delta_{ij} E^2)
 \end{aligned}$$

Define Maxwell's stress tensor

$$T_{ij} = \frac{1}{4\pi} [E_i E_j + B_i B_j - \frac{1}{2} \delta_{ij} (E^2 + B^2)] \quad \left(\begin{array}{l} \text{note } T_{ij} = T_{ji} \\ \text{Symmetric tensor} \end{array} \right)$$

Then

$$\begin{aligned}
 \frac{d}{dt} \vec{p}_i^{\text{mech}} + \frac{d}{dt} \int_V d^3r \Pi_i &= \int_V d^3r \partial_j T_{ij} & \left(\partial_j T_{ij} = \frac{\partial T_{ij}}{\partial x_j} \right) \\
 &= \oint_S da T_{ij} \hat{m}_j \\
 \frac{d}{dt} \vec{p}^{\text{mech}} + \frac{d}{dt} \int_V d^3r \vec{\Pi} &= \oint_S da \vec{T} \cdot \hat{m}
 \end{aligned}$$

- T_{ij} gives the flow of the i th component of electromagnetic field momentum through an element of surface area \perp to direction \hat{e}_j

~~For static situations where $\frac{d\Pi}{dt} = 0$, $\frac{d\vec{p}^{\text{mech}}}{dt} = \vec{F} = \oint_S da \vec{T} \cdot \hat{m}$
Gives electromagnetic force on the surface S~~

Note: $\frac{d\vec{P}^{\text{mech}}}{dt}$ is ~~also~~ equal to the total electromagnetic force on the volume V .

Hence we can write

$$\vec{F}_{\text{EM}} = \oint_S da \vec{T} \cdot \hat{n} - \frac{d}{dt} \int_V d^3r \vec{T}$$

for static situations, the 2nd term vanishes and

$$\vec{F}_{\text{EM}} = \oint_S da \vec{T} \cdot \hat{n} \quad \vec{T}_{ij} \text{ is } i\text{th component of static force on unit area with normal } \hat{e}_j$$

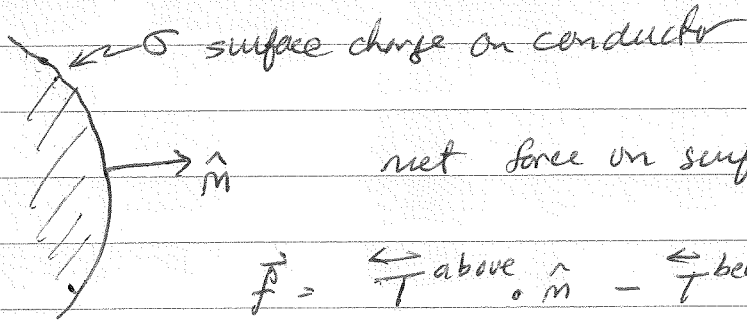
this is origin of the term "stress" tensors.

\vec{T} is like the stress tensor of an elastic medium.

T_{xx}, T_{yy}, T_{zz} are like pressure.

off diagonal elements are like shear stresses

Force on a conductor surface



net force on surface per unit area is

$$\vec{f} = \vec{T}_{\text{above}} \cdot \hat{n} - \vec{T}_{\text{below}} \cdot \hat{n}$$

$\uparrow = 0$ as $\vec{E} = 0$ inside conductor

$$\vec{f} = \frac{1}{4\pi} \left[\vec{E} (\vec{E} \cdot \hat{n}) - \frac{1}{2} \hat{n} E^2 \right]$$

for conducting surface

$$\hat{n} \cdot \vec{E}^{\text{above}} = 4\pi\sigma \quad (\text{since } \vec{E}^{\text{below}} = 0)$$

and tangential component $\vec{E} = 0$

$$\Rightarrow \vec{E} = 4\pi\sigma \hat{n}$$

$$\text{So } \vec{f} = \frac{1}{4\pi} \left[(4\pi\sigma \hat{n})(4\pi\sigma) - \frac{1}{2} \hat{n} (4\pi\sigma)^2 \right]$$

$$\vec{f} = \frac{\hat{n}}{4\pi} \left[(4\pi\sigma)^2 - \frac{1}{2} (4\pi\sigma)^2 \right]$$

$$\vec{f} = \frac{\hat{n}}{4\pi} \left[(4\pi\sigma)^2 - \frac{1}{2} (4\pi\sigma)^2 \right] = 2\pi\sigma^2 \hat{n}$$

force per unit area

$$\vec{f} = 2\pi\sigma^2 \hat{n} = \frac{1}{2} \sigma \vec{E}$$

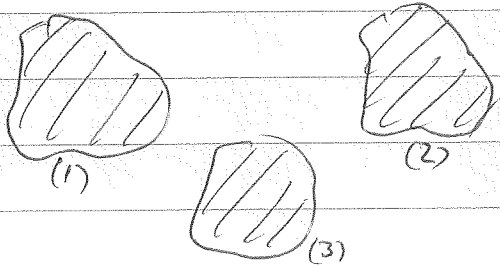
$$\vec{f} = \sigma \vec{E}_{\text{ave}}$$

where $\vec{E}_{\text{ave}} = \frac{1}{2} (\vec{E}^{\text{above}} + \vec{E}^{\text{below}})$
is average field at surface
averaging over above + below

↑
Note factor $\frac{1}{2}$. Naively one might have thought $\vec{f} = \sigma \vec{E}$. But need to exclude self field of charge on surface from acting on itself. See also Jackson pg 42 for another approach.

Capacitance

Consider a set of conductors with potential $\phi(\vec{r}) = V_i$ fixed on conductor i



(also need condition on $\phi(\vec{r}) \rightarrow \infty$ if system is not enclosed)

From uniqueness theorem we know that specifying the V_i on each conductor is enough to determine the potential $\phi(\vec{r})$ everywhere. We can write this potential in the following form -

Let $\phi^{(i)}(\vec{r})$ be the solution to the boundary value problem $\nabla^2 \phi^{(i)}(\vec{r}) = 0$ and $\phi^{(i)}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \text{ on surface of conductor } (i) \\ 0 & \text{if } \vec{r} \text{ on surface of any other conductor } (j), j \neq i \end{cases}$

Then by superposition

$$\phi(\vec{r}) = \sum_i V_i \phi^{(i)}(\vec{r})$$

is solution to the problem $\nabla^2 \phi = 0$ and $\phi(\vec{r}) = V_i$ for \vec{r} on surface of conductor (i)

The surface charge density at \vec{r} on surface of conductor (i) is

$$\sigma^{(i)}(\vec{r}) = \frac{-1}{4\pi} \frac{\partial \phi(\vec{r})}{\partial m} = -\frac{1}{4\pi} \sum_j V_j \frac{\partial \phi^{(j)}(\vec{r})}{\partial m}$$

where $\frac{\partial \phi}{\partial m} = (\vec{\nabla} \phi) \cdot \hat{m}$ is the derivative normal to the surface at point \vec{r} .

The total charge on conductor (i) is

$$Q_i = \int_{S_i} da \sigma^{(i)}(\vec{r}) = -\frac{1}{4\pi} \sum_j V_j \int_{S_i} da \frac{\partial \phi^{(j)}}{\partial n}$$

↑
surface of conductor (i)

Define $C_{ij} \equiv -\frac{1}{4\pi} \int_{S_i} da \frac{\partial \phi^{(j)}}{\partial n}$

The C_{ij} depend only on the geometry of the conductors

Then we have

$$Q_i = \sum_j C_{ij} V_j$$

C_{ij} is the capacitance matrix

The charge on conductor (i) is a linear function of the potentials V_j on the conductors (j)

Since we know that specifying the Q_i that is on each conductor will uniquely determine $\phi(\vec{r})$ and hence the potential V_i on each conductor, the capacitance matrix is invertible

$$V_i = \sum_j [C^{-1}]_{ij} Q_j$$

The electrostatic energy of the conductors is then

$$E = \frac{1}{2} \int d^3r \rho \phi = \frac{1}{2} \sum_i Q_i V_i = \frac{1}{2} \sum_{i,j} C_{ij} V_i V_j = \frac{1}{2} \sum_{i,j} C_{ij}^{-1} Q_i Q_j$$

$[V \cdot C \cdot V]$ $[Q \cdot C^{-1} \cdot Q]$

Common to define capacitance of two conductors by

$$C = \frac{Q}{V_1 - V_2}$$

when conductor (1) has charge Q
conductor (2) has charge $-Q$
 $V_1 - V_2$ is potential difference
between the two conductors.

all other conductors fixed at $V_i = 0$

We can determine C in terms of the elements of the matrix C_{ij}

$$\begin{cases} Q = C_{11}V_1 + C_{12}V_2 \\ -Q = C_{21}V_1 + C_{22}V_2 \end{cases} \Rightarrow V_2 = -\left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}}\right)V_1$$

$$\Rightarrow Q = \left[C_{11} - C_{12} \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right) \right] V_1$$

$$V_1 - V_2 = \left[1 + \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right) \right] V_1$$

$$C = \frac{Q}{V_1 - V_2} = \frac{C_{11} - C_{12} \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right)}{1 + \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right)}$$

$$C = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11} + C_{12} + C_{21} + C_{22}}$$

Capacitance can also be defined when the space between the conductors is filled with a dielectric ϵ . In this case, if Q_f is the free charge, then Q_f/ϵ is the effective total charge to use in computing ϕ .

$$\Rightarrow \frac{Q_i}{\epsilon} = \sum_j C_{ij}^{(0)} V_j$$

where $C_{ij}^{(0)}$ are capacitances appropriate to a vacuum between the conductors

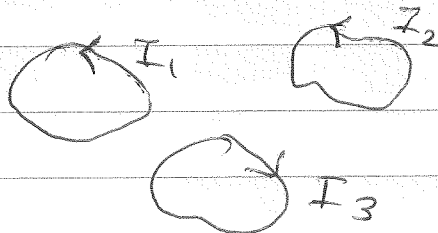
$$\Rightarrow Q_i = \sum_j \epsilon C_{ij}^{(0)} V_j$$

$$= \sum_j C_{ij} V_j \quad \text{where } C_{ij} = \epsilon C_{ij}^{(0)}$$

the capacitance is increased by a factor the dielectric constant ϵ .

Inductance

Consider a set of current carrying loops C_i with currents I_i



In Coulomb gauge, we can write the magnetic vector potential \vec{A} from these current loops as

$$\vec{A}(\vec{r}) = \frac{1}{c} \int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \sum_i \frac{I_i}{c} \oint_{C_i} \frac{d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

↑ integrate over loop C_i
integration variable is \vec{r}'

The magnetic flux through loop i is

$$\Phi_i = \int_{S_i} da \hat{n} \cdot \vec{B} = \int_{S_i} da \hat{n} \cdot \vec{\nabla} \times \vec{A} = \oint_{C_i} d\vec{l} \cdot \vec{A}$$

↑ surface bounded by loop C_i

$$\Phi_i = \sum_j \frac{I_j}{c} \oint_{C_i} \oint_{C_j} \frac{d\vec{l}_i \cdot d\vec{l}_j}{|\vec{r}_i - \vec{r}_j|}$$

pure geometrical quantity

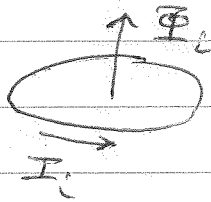
$$\Phi_i = c \sum_j M_{ij} I_j$$

$$\text{where } M_{ij} = \oint_{C_i} \oint_{C_j} \frac{d\vec{l}_i \cdot d\vec{l}_j}{c^2 |\vec{r}_i - \vec{r}_j|}$$

is the mutual inductance of loops (i) and (j) . $M_{ij} = M_{ji}$

$L_i \equiv M_{ii}$ is self-inductance of loop (i)

The sign convention in the above is that, Φ_i is computed in direction given by right hand rule, according to the direction taken for current in loop (i)



Magnetostatic energy

$$\mathcal{E} = \frac{1}{2c} \int d^3r \vec{j} \cdot \vec{A} = \frac{1}{2c} \sum_i \oint d\vec{l} \cdot \vec{A} I_i$$

$$= \frac{1}{2c} \sum_i \Phi_i I_i$$

$$\mathcal{E} = \frac{1}{2} \sum_{i,j} M_{ij} I_i I_j$$