$\chi(t)$ is the response to $\vec{E}(t) = \delta(t)$

For our simple model

$$\tilde{\alpha}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \frac{e^2}{m}\, \frac{1}{\omega_0^2-\omega^2-i\omega\gamma}$$

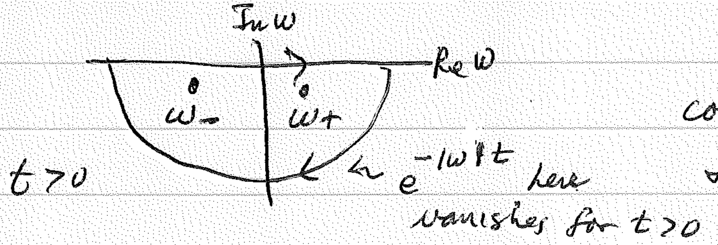
do by contour integration

$$\frac{1}{\omega^2+i\gamma\omega-\omega_0^2} = \frac{1}{(\omega-\omega_+)(\omega-\omega_-)}$$

$$\omega_\pm = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2-\gamma^2/4} = -\frac{i\gamma}{2} \pm \bar{\omega}$$

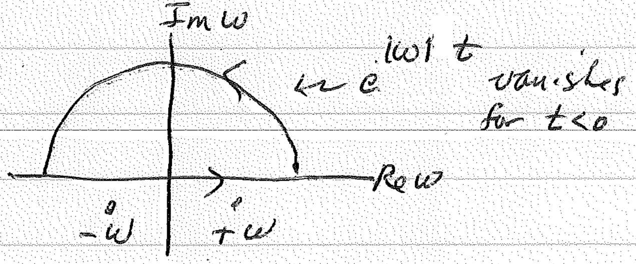
poles at $\omega_\pm$ are in lower half complex plane.

for $t>0$, close contour in lower half plane



contour encloses poles + get contribution

for $t<0$, close contour in upper half plane



contour encloses no poles ⇒ integral vanishes

$\chi(t) = 0$ for $t<0$

<u>causal response</u>! No polarization until electric field turns on

For $t > 0$

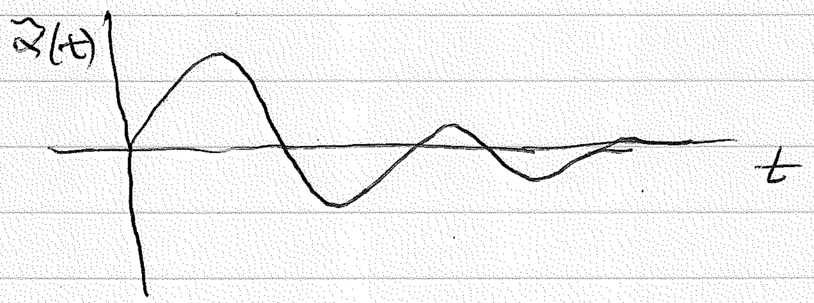
$$\tilde{\alpha}(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \frac{e^2}{m}\, \frac{(-1)}{(\omega-\omega_+)(\omega-\omega_-)}$$

$$= \underset{\nearrow}{(-2\pi i)}\, \frac{e^2}{m}\, \frac{(-1)}{2\pi} \left[\frac{e^{-i\omega_+ t}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_- t}}{\omega_- - \omega_+} \right]$$

from Residue theorem

$$= \frac{i e^2}{m} \left[\frac{e^{-\gamma t/2} e^{-i\bar{\omega} t}}{2\bar{\omega}} - \frac{e^{-\gamma t/2} e^{i\bar{\omega} t}}{2\bar{\omega}} \right]$$

$$\tilde{\alpha}(t) = \begin{cases} \dfrac{e^2}{m} \dfrac{e^{-\gamma t/2}}{\bar{\omega}} \sin(\bar{\omega} t) & t > 0 \\ 0 & t < 0 \end{cases}$$



damped oscillation

Polarization density $\vec{\mathbb{P}}_\omega = 4\pi \chi(\omega) \vec{E}_\omega$ for harmonic oscillation

$$\chi(\omega) \approx \underset{\uparrow \text{ atom density}}{n}\, \alpha(\omega)$$ for dilute system

can use Clausius - Mossotte correction for denser materials

$$\Rightarrow \vec{D}_\omega = \underset{\uparrow \text{ freq dependent}}{\varepsilon(\omega)}\, \vec{E}_\omega \qquad \varepsilon(\omega) = 1 + 4\pi\chi(\omega)$$

→ as with $\vec{P}$ and $\vec{E}$, relation between $\vec{D}$ and $\vec{E}$ is <u>non-local</u> in time

$$\vec{D}(t) \neq \varepsilon \vec{E}(t)$$

rather

$$\vec{D}(t) = \int_{-\infty}^{\infty} dt' \, \vec{E}(t') \, \tilde{\varepsilon}(t-t')$$

↑ Fourier transf of $\varepsilon(\omega)$

Ampere's law is

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

becomes $$\frac{1}{\mu} \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\int_{-\infty}^{\infty} dt' \, \vec{E}(t') \frac{d}{dt}\tilde{\varepsilon}(t-t')$$

↑ integro-differential equation!

Maxwells equations only look simple when expressed in terms of Fourier transforms

$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{B}(\vec{r},t) = \vec{B}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{D}(\vec{r},t) = \vec{D}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{H}(\vec{r},t) = \vec{H}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Maxwell's Equ for <u>source free</u> system $\rho = \vec{j} = 0$

$$\vec{\nabla}\cdot\vec{D} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{c\,\partial t}, \quad \vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{c\,\partial t}$$

assume μ is true constant - not freq dependent

dielectric response is $\vec D_\omega = \varepsilon(\omega)\vec E_\omega$

Then for the Fourier amplitudes of the fields, Maxwell's Equations become

transverse polarized

1) $i\vec k\cdot\vec D_\omega = i\,\varepsilon(\omega)\,\vec k\cdot\vec E_\omega = 0 \quad\Rightarrow\quad \boxed{\vec k\perp\vec E_\omega}$ (unless $\varepsilon(\omega)=0$)

2) $i\vec k\cdot\vec B_\omega = 0 \quad\Rightarrow\quad \boxed{\vec k\perp\vec B_\omega}$

3) $i\vec k\times\vec E_\omega = i\frac{\omega}{c}\vec B_\omega$

4) $i\vec k\times\vec H_\omega = -i\frac{\omega}{c}\vec D_\omega \quad\Rightarrow\quad \frac{i\vec k}{\mu}\times\vec B_\omega = -\frac{i\omega}{c}\varepsilon(\omega)\vec E_\omega$

$$\vec k\times(3) = i\vec k\times(\vec k\times\vec E_\omega) = i\frac{\omega}{c}\vec k\times\vec B_\omega$$

$$\Rightarrow\quad -ik^2\vec E_\omega = -\frac{i\omega^2}{c^2}\varepsilon(\omega)\mu\,\vec E_\omega$$ using (4)

$$\boxed{k^2 = \frac{\omega^2}{c^2}\varepsilon(\omega)\mu}$$ <u>dispersion relation</u>

$$\cancel{\vec B_\omega = \frac{\vec k}{\omega}\times\vec E_\omega = \sqrt{\varepsilon(\omega)\mu}\,\hat k\times\vec E_\omega}$$

<u>Note</u>: $\frac{\omega}{|k|} = \frac{c}{\sqrt{\varepsilon(\omega)\mu}}$ varies with ω.

There is not a single phase velocity.

$\Rightarrow \vec E$ is <u>not</u> in general a solution of a wave equation - different frequencies travel with different speeds

Since $\varepsilon(\omega)$ is complex $\qquad \varepsilon(\omega) = \varepsilon_1(\omega) + i\,\varepsilon_2(\omega)$

⇒ wave vector also complex $\qquad$ For $\vec{k} = k\,\hat{z}$

$$k = k_1 + i k_2 = \pm \frac{\omega}{c}\sqrt{\mu}\,\sqrt{\varepsilon_1 + i\,\varepsilon_2}$$

$$\vec{E}(\vec{r},t) = \vec{E}_\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{E}_\omega\, e^{i[(k_1+ik_2)z-\omega t]}$$
$$= \vec{E}_\omega\, e^{-k_2 z}\, e^{i(k_1 z-\omega t)}$$

k_1 determines the oscillation of the wave

k_2 determines the decay or attenuation of the wave as it propagates into the material

phase velocity $\qquad v_p \equiv \dfrac{\omega}{k_1}$

index of refraction $\qquad n = \dfrac{c}{v_p} = \dfrac{c k_1}{\omega}$

group velocity $\quad v_g \qquad \dfrac{1}{v_g} = \dfrac{dk_1}{d\omega}$

Magnetic field: $\quad \vec{B}_\omega = \dfrac{c\vec{k}}{\omega} \times \vec{E}_\omega$

for $\vec{k} = k\,\hat{z}$, $\quad \vec{B}_\omega = \dfrac{c(k_1+ik_2)}{\omega}\,\hat{z}\times\vec{E}_\omega$

if $\quad k_1 + i k_2 = \sqrt{k_1^2+k_2^2}\; e^{i\delta} \qquad \delta = \arctan\left(\dfrac{k_2}{k_1}\right)$

$$= |k|\,e^{i\delta}$$

$$\vec{B}_\omega = \frac{c|k|}{\omega}\,\hat{z}\times\vec{E}_\omega\,\underset{\uparrow\ \text{phase shift}}{e^{i\delta}}$$

$$\vec{B}(\vec{r},t) = \frac{c|k|}{\omega}(\hat{z}\times\vec{E}_\omega)\, e^{-k_2 z}\, e^{i(k_1 z - \omega t + \delta)}$$

Physical fields - take real parts

$$\vec{E}(\vec{r},t) = \vec{E}_\omega\, e^{-k_2 z}\cos(k_1 z - \omega t)$$

$$\vec{B}(\vec{r},t) = (\hat{z}\times\vec{E}_\omega)\frac{c|k|}{\omega}\, e^{-k_2 z}\cos(k_1 z - \omega t + \delta)$$

Conclusions

1) $\vec{E}$ and $\vec{B}$ $\perp$ $\vec{k}$ transverse polarized

2) $\vec{E} \perp \vec{B}$

3) amplitude ratio $\frac{|\vec{B}|}{|\vec{E}|} = \frac{c|k|}{\omega} = \sqrt{|\varepsilon(\omega)|\mu}$

4) $\vec{B}$ is shifted in phase with respect to $\vec{E}$ by phase shift $\delta = \arctan(k_2/k_1)$

5) waves decay as they propagate $e^{-k_2 z}$

consequence of complex $\varepsilon(\omega)$

If $\varepsilon_2 = 0$, ie $\varepsilon(\omega)$ is real, and if $\varepsilon > 0$, then $k_2 = 0 \Rightarrow$ no decay, no phase shift

consequences of frequency dependence of $\varepsilon(\omega)$

6) $\vec{E}(t)$ and $\vec{D}(t)$ non locally related in time

7) waves of different ω travel with different $v_p = \omega/k_1$

8) dispersion - wave pulses do not travel with v_p and they spread as the propagate

pulses travel with group velocity $v_g = \frac{d\omega}{dk}$
(see Quantum Mechanics discussion)

$v_g < v_p$ "normal dispersion"

$v_g > v_p$ "anomalous dispersion"

$$\frac{1}{v_g} = \frac{dk_1}{d\omega} = \frac{d}{d\omega}\left[\frac{\omega}{c}\overset{\text{index of refraction}}{n}\right]$$

$$\frac{1}{v_g} = \frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega} = \frac{1}{v_p} + \frac{\omega}{c}\frac{dn}{d\omega}$$

$$v_g = \frac{v_p}{1 + \frac{v_p}{c}\,\omega\frac{dn}{d\omega}}$$

$$\Rightarrow \text{ when } \begin{cases} \frac{dn}{d\omega} > 0, & v_g < v_p \quad \text{normal dispersion} \\ \frac{dn}{d\omega} < 0, & v_g > v_p \quad \text{anomalous dispersion} \end{cases}$$