We will see that this situation in general corresponds to an *elliptically polarized* wave!

<u>General case</u> E_1 and E_2 are complex constants

write $E_1 \hat{e}_1 + E_2 \hat{e}_2 \equiv \vec{u}\, e^{i\psi}$

where ψ is chosen so that $\vec{u}\cdot\vec{u}$ is <u>real</u>

– one can always do this since $\vec{u}\cdot\vec{u} = (E_1^2 + E_2^2)\, e^{-2i\psi}$

so 2ψ is just the phase of the complex $E_1^2 + E_2^2$

$\vec{u}$ is a complex vector $\Rightarrow$ $\vec{u} = \vec{u}_a + i\,\vec{u}_b$

with $\vec{u}_a$ and $\vec{u}_b$ real vectors

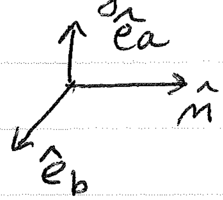
Since $\vec{u}\cdot\vec{u}$ is real $\Rightarrow$ $\vec{u}_a \cdot \vec{u}_b = 0$

so $\vec{u}_a \perp \vec{u}_b$ orthogonal

Let $\hat{e}_a$ be the unit vector in direction of $\vec{u}_a$

so $\vec{u}_a = u_a \hat{e}_a$ with $u_a = |\vec{u}_a|$

Let $\hat{e}_b = \hat{n} \times \hat{e}_a$ so that $\{\hat{n}, \hat{e}_a, \hat{e}_b\}$ are a right handed coordinate system



Then $\vec{u}_b = \pm u_b \hat{e}_b$ where $u_b = |\vec{u}_b|$

since $\vec{u}_b \perp \vec{u}_a$ and both are $\perp$ to $\hat{n}$.

It is $(+)$ if $\vec{u}_b$ is parallel to $\hat{e}_b$ and it is $(-)$ if $\vec{u}_b$ is antiparallel to $\hat{e}_b$.

In this representation we have

$$\vec{E}(\vec{r},t) = Re\left\{ \vec{U} e^{i\psi} e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$

$$= e^{-k_2 \hat{n}\cdot\vec{r}} Re\left\{ U_a \hat{e}_a e^{i(k_1 \hat{n}\cdot\vec{r} - \omega t + \psi)} \pm U_b \hat{e}_b (+i) e^{i(k_1 \hat{n}\cdot\vec{r} - \omega t + \psi)} \right\}$$

$$= e^{-k_2 \hat{n}\cdot\vec{r}} \left\{ U_a \hat{e}_a \cos(\Phi + \psi) \mp U_b \hat{e}_b \sin(\Phi + \psi) \right\}$$

where we write $\Phi \equiv k_1 \hat{n}\cdot\vec{r} - \omega t$

Let's define $e^{-k_2 \hat{n}\cdot\vec{r}} U_a \rightarrow U_a$

$e^{-k_2 \hat{n}\cdot\vec{r}} U_b \rightarrow U_b$

so we don't have to keep writing the constant attenuation factor that is a common factor of all components of $\vec{E}$.

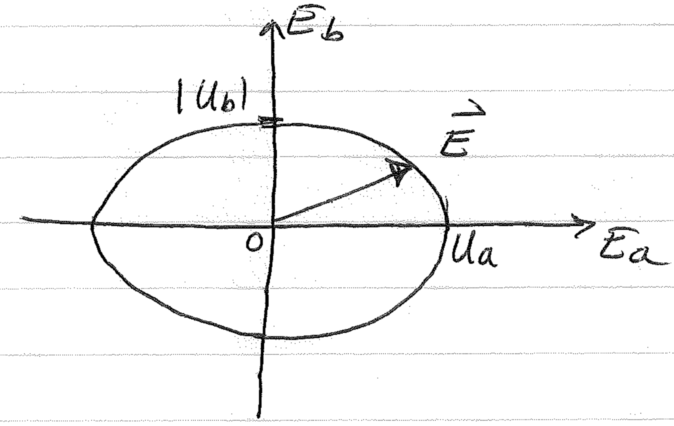
Then define E_a and E_b as the components of $\vec{E}$ in the directions $\hat{e}_a$ ad $\hat{e}_b$ respectively.

$$E_a = U_a \cos(\Phi + \psi)$$
$$E_b = \mp U_b \sin(\Phi + \psi)$$

This then gives

$$\left(\frac{E_a}{U_a}\right)^2 + \left(\frac{E_b}{U_b}\right)^2 = \cos^2(\Phi + \psi) + \sin^2(\Phi + \psi) = 1$$

This is just the equation for an ellipse with semiaxes of lengths U_a and U_b, oriented in the directions of $\hat{e}_a$ and $\hat{e}_b$.



$\Rightarrow$ At a fixed position $\vec{r}$, the tip of the vector $\vec{E}$ will trace out the above ellipse as the time increases by one period of oscillation $2\pi/\omega$.

For (+), ie $\vec{U}_b = U_b \hat{e}_b$, $\vec{E}$ goes around the ellipse <u>counterclockwise</u> as t increases

For (−), ie $\vec{U}_b = -U_b \hat{e}_b$, $\vec{E}$ goes around the ellipse <u>clockwise</u> as t increases

Such a wave is said to be <u>elliptically polarized</u>

<u>Special cases</u>

① $U_a = 0$ or $U_b = 0$
the wave is <u>linearly polarized</u>

(2) $u_a = u_b$

The tip of $\vec{E}$ traces out a ~~circl~~ circle as t increases. The wave is circularly polarized.

The (+) case is said to have right handed circular polarization

the (−) case is said to have left handed circular polarization

One can define circular polarization basis vectors

$$\hat{e}_+ \equiv \frac{\hat{e}_a + i\hat{e}_b}{\sqrt{2}} \qquad \hat{e}_- \equiv \frac{\hat{e}_a - i\hat{e}_b}{\sqrt{2}}$$

with $\hat{e}_a$ and $\hat{e}_b$ orthogonal.

A wave with complex amplitude $\vec{E}_\omega = E\,\hat{e}_+$ is right handed circularly polarized.

A wave with complex amplitude $\vec{E}_\omega = E\,\hat{e}_-$ is left handed circularly polarized.

Just as the general case can always be written as a superposition of two orthogonal linearly polarized waves, ie

$$\vec{E}_\omega = E_1\,\hat{e}_1 + E_2\,\hat{e}_2$$

One can also always write the general case as a superposition of a left handed and a right handed circularly polarized wave

$$\vec{U} = \vec{U}_a + i\vec{U}_b = U_a \hat{e}_a \pm i U_b \hat{e}_b$$

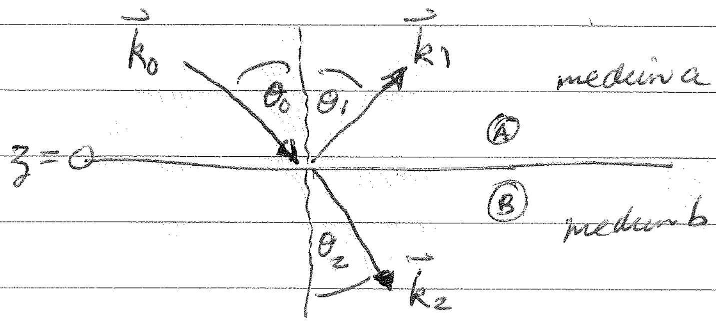
$$= \left(\frac{U_a + U_b}{\sqrt{2}}\right) \hat{e}_\pm + \left(\frac{U_a - U_b}{\sqrt{2}}\right) \hat{e}_\mp$$

(~~rexpand~~ substitute in for $\hat{e}_\pm$ and expand, to see that this is so)

$\Rightarrow$ An elliptically polarized wave can be written as a superposition of circularly polarized waves

As a special case of the above (if $U_a = 0$ or $U_b = 0$) a linearly polarized wave can always be written as a superposition of circularly polarized waves.

Reflection + Transmission of Waves at Interfaces



consider wave propagating from medium A into medium B.

for simplicity assume ϵ_a is real and positive, ϵ_b may be complex
μ_a and μ_b are real and constant

$\vec{k}_0$ is incident wave, θ_0 = angle of incidence
$\vec{k}_1$ is reflected wave, θ_1 = angle of reflection
$\vec{k}_2$ is the transmitted or "refracted" wave, θ_2 = angle of refraction

let each wave be given by

$$\vec{F}_n(\vec{r},t) = \vec{F}_n \, e^{i(\vec{k}_n \cdot \vec{r} - \omega_n t)}$$

where $\vec{F}_n$ can be either $\vec{E}_n$ or $\vec{H}_n$ for the electric or magnetic component of the wave

boundary condition: tangential component $\vec{E}$ must be continuous at $z=0$. If $\hat{t}$ is a vector in xy plane, and we consider $\vec{r}=0$, then

$$\Rightarrow \hat{t} \cdot \vec{E}_0 e^{-i\omega_0 t} + \hat{t} \cdot \vec{E}_1 e^{-i\omega_1 t} = \hat{t} \cdot \vec{E}_2 e^{-i\omega_2 t}$$

must be true for all time. Can only happen if

$$\boxed{\omega_0 = \omega_1 = \omega_2 \equiv \omega}$$ all frequencies are equal

Now consider the same boundary condition for $\vec{\rho}$ a position vector in the xy plane at $z=0$. Since ω's all equal we can cancel out the common $e^{-i\omega t}$ factors to get

$$\hat{t}\cdot\vec{E}_0\, e^{i\vec{k}_0\cdot\vec{\rho}} + \hat{t}\cdot\vec{E}_1\, e^{i\vec{k}_1\cdot\vec{\rho}} = \hat{t}\cdot\vec{E}_2\, e^{i\vec{k}_2\cdot\vec{\rho}}$$

this must be true for all $\vec{\rho}$. Can only happen if the projections of the $\vec{k}_n$ in the xy plane are all equal

$$\boxed{\begin{aligned} k_{0x} &= k_{1x} = k_{2x} \\ k_{0y} &= k_{1y} = k_{2y} \end{aligned}}$$

only z components $\vec{k}$ vectors can be different

choose coord system as in diagram so that all $\vec{k}$ vectors lie in the xz plane (y is out of page)

Since ϵ_a is real and positive, $\vec{k}_0$ and $\vec{k}_1$ are real vectors

$$k_{0x} = k_{1x} \Rightarrow |\vec{k}_0|\sin\theta_0 = |\vec{k}_1|\sin\theta_1$$

since $k_0^2 = \frac{\omega^2}{c^2}\mu_a\epsilon_a$ and $k_1^2 = \frac{\omega^2}{c^2}\mu_a\epsilon_a$

then $|\vec{k}_0| = |\vec{k}_1|$ so $\sin\theta_0 = \sin\theta_1$

$$\boxed{\theta_0 = \theta_1}$$

angle of incidence = angle of reflection

If ϵ_b is also real and positive (B is transparent) then $|\vec{k}_2|$ is real

$$k_{0x} = k_{2x} \Rightarrow |\vec{k}_0| \sin\theta_0 = |\vec{k}_2| \sin\theta_2$$

$$k_2^2 = \frac{\omega^2}{c^2} \mu_b \epsilon_b$$

$$\Rightarrow \sqrt{\mu_a \epsilon_a} \sin\theta_0 = \sqrt{\mu_b \epsilon_b} \sin\theta_2$$

in terms of index of refraction $n = \frac{kc}{\omega} = \frac{\omega\sqrt{\mu\epsilon}}{c} \frac{c}{\omega}$

$$n = \sqrt{\mu\epsilon}$$

$$\Rightarrow n_a \sin\theta_0 = n_b \sin\theta_2$$

$$\boxed{\frac{\sin\theta_2}{\sin\theta_0} = \frac{n_a}{n_b}}$$

Snell's Law true for all types of waves, not just EM waves

If $n_a > n_b$ then $\theta_2 > \theta_0$

In this case, when θ_0 is too large, we will have $\frac{n_a}{n_b} \sin\theta_0 > 1$ and there will be no solution for θ_2

$\Rightarrow$ no transmitted wave

This is "total internal reflection" – wave does not exit medium A. The critical angle, above which one has total internal reflection, is given by

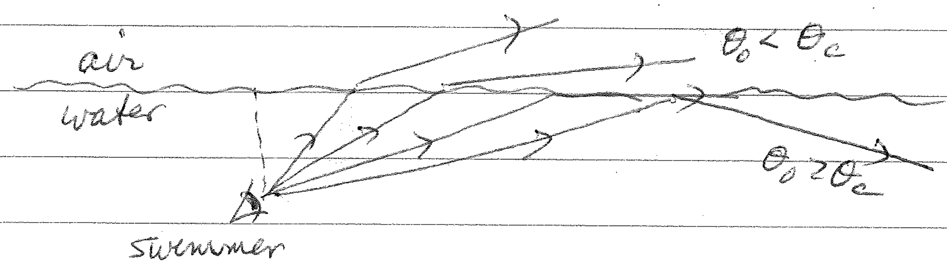
$$\frac{n_a}{n_b} \sin\theta_c = 1 \quad , \quad \boxed{\theta_c = \arcsin\left(\frac{n_b}{n_a}\right)}$$

$\epsilon \sim 1 + 4\pi N \alpha$ density

Since $n = \sqrt{\mu\epsilon}$ and ϵ grows with density of the material, one usually has total internal reflection when one goes from a denser to a less dense medium.

<u>Examples</u>. diamonds sparkle due to total internal reflection. Diamonds have large $n \Rightarrow$ small θ_c $\Rightarrow$ light bounces around inside many times before it can exit

Can also see total internal reflection when swimming under water



<u>More general case</u> $\sqrt{\epsilon_b}$ is complex so $\vec{k}_2$ is complex

$$\vec{k}_2 = \vec{k}_2' + i\vec{k}_2'' \qquad k_2' = |\vec{k}_2'|$$

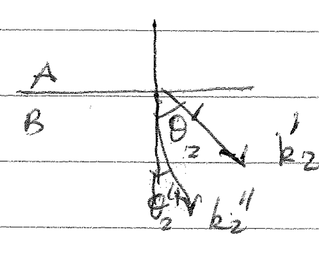
$$k_2'' = |\vec{k}_2''|$$

real part. imaginary part

Note $\vec{k}_2'$ and $\vec{k}_2''$ need not be in the same direction!

Condition $k_{0x} = k_{2x} \Rightarrow \begin{cases} k_{0x} = k_{2x}' \\ 0 = k_{2x}'' \end{cases}$ equate real and imaginary parts

$$k_0 \sin\theta_0 = k_2' \sin\theta_2'$$
$$0 = k_2'' \sin\theta_2''$$



$\Rightarrow \underline{\theta_2''} = 0$
$\vec{k}_2'' = k_2'' \hat{z}$ } attenuation factor for the transmitted wave is $e^{-k_2'' z}$

$\Rightarrow$ planes of constant amplitude are always parallel to the interface no matter what the angle of incidence θ_0

Having found θ_2'' there are still three quantities we must yet find in order to characterize the transmitted wave. These are θ_2', k_2', k_2''.

To solve for these we will need 3 equations

one is: $\quad k_0 \sin\theta_0 = k_2' \sin\theta_2' \qquad (1)$
(from boundary condition)

where $k_0 = \frac{\omega}{c}\sqrt{\mu_a \epsilon_a} = \frac{\omega}{c} n_a$ — dispersion relation in medium a

The other two come from equating the real and imaginary parts of the dispersion relation in medium b.

$$k_2^2 = \frac{\omega^2}{c^2}\mu_b \epsilon_b = \frac{\omega^2}{c^2}\mu_b(\epsilon_{b1} + i\epsilon_{b2})$$

$$k_2^2 = (\vec{k}_2' + i\vec{k}_2'')\cdot(\vec{k}_2' + i\vec{k}_2'')$$
$$= (k_2')^2 - (k_2'')^2 + 2i\,\vec{k}_2'\cdot\vec{k}_2''$$
$$= (k_2')^2 - (k_2'')^2 + 2i\,k_2' k_2'' \cos\theta_2'$$

equate real and imaginary parts

$$(k_2')^2 - (k_2'')^2 = \frac{\omega^2}{c^2}\mu_b \varepsilon_{b1} \qquad (2)$$

$$2 k_2' k_2'' \cos\theta_2' = \frac{\omega^2}{c^2}\mu_b \varepsilon_{b2} \qquad (3)$$

Use (2) and (3) to solve for k_2' and k_2'' in terms of θ_2'

$$(2) \Rightarrow (k_2')^2 = (k_2'')^2 + \frac{\omega^2}{c^2}\mu_b \varepsilon_{b1} \qquad (4)$$

$$(3) \Rightarrow k_2'' = \frac{\omega^2}{c^2}\frac{\mu_b \varepsilon_{b2}}{2k_2' \cos\theta_2'} \qquad (5)$$

plug (5) into (4)

$$(k_2')^2 = \left(\frac{\omega^2}{c^2}\frac{\mu_b \varepsilon_{b2}}{2k_2'\cos\theta_2'}\right)^2 + \frac{\omega^2}{c^2}\mu_b\varepsilon_{b1}$$

$$\Rightarrow (k_2')^4 - \frac{\omega^2}{c^2}\mu_b\varepsilon_{b1}(k_2')^2 - \frac{\omega^4}{c^4}\frac{\mu_b^2\varepsilon_{b2}^2}{4\cos^2\theta_2'} = 0$$

solve quadratic formula

$$(k_2')^2 = \frac{\omega^2 \mu_b \varepsilon_{b1}}{2c^2} + \sqrt{\frac{\omega^4 \mu_b^2 \varepsilon_{b1}^2}{4c} + \frac{\omega^4 \mu_b^2 \varepsilon_{b2}^2}{4c^4\cos^2\theta_2'}}$$

take (+) solution only since $(k_2')^2$ must be positive

$$= \frac{\omega^2 \mu_b}{c^2}\left[\frac{\varepsilon_{b1}}{2} + \frac{1}{2}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_2'}}\right]$$

(6) $$\boxed{k_2' = \frac{\omega}{c}\sqrt{\mu_b}\left[\frac{1}{2}\epsilon_{b1} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\right]^{1/2}}$$

then get k_2'' from (4)

$$(k_2'')^2 = (k_2')^2 - \frac{\omega^2}{c^2}\mu_b\varepsilon_{b1}$$

(7) $$\boxed{k_2'' = \frac{\omega}{c}\sqrt{\mu_b}\left[-\frac{1}{2}\epsilon_{b1} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\right]^{1/2}}$$

Note, these reduce to what we found earlier for the real and imaginary parts of the wave vector for a plane wave in a medium with complex ε, IF we take $\theta_2' = 0$. We will have $\theta_2' = 0$ for normal incidence $\theta_0 = 0$.

Both k_2' and k_2'' above still depend on the angle of refraction θ_2'. We can close the set of equation by adding in Eq (1)

$$k_0 \sin\theta_0 = k_2' \sin\theta_2'$$

(8) or $$\frac{\omega}{c} n_a \sin\theta_0 = k_2' \sin\theta_2'$$

where $n_a = \frac{k_0 c}{\omega} = \sqrt{\mu_a \varepsilon_a}$

Since the pair of equations (6) and (8) only involve the unknowns k_1' and θ_2' we can