use them to eliminate k_2' and get a final single equation that determines θ_2')

Define index of refraction in medium b

$$n_b = \sqrt{\mu_b \varepsilon_{b1}}$$

Then

$$\frac{\omega}{c} n_a \sin\theta_0 = \frac{\omega}{c} n_b \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\varepsilon_{b2}^2}{\varepsilon_{b1}^2 \cos^2\theta_2'}}\right]^{1/2} \sin\theta_2'$$

or

$$\boxed{n_a \sin\theta_0 = n_b \sin\theta_2' \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\varepsilon_{b2}^2}{\varepsilon_{b1}^2 \cos^2\theta_2'}}\right]^{1/2}}$$

This is the analog of Snell's law for propagation into a medium with complex dielectric function ε

Cases

① For a nearly transparent material with $\varepsilon_{b2} \ll \varepsilon_{b1}$ we can expand in $\frac{\varepsilon_{b2}}{\varepsilon_{b1}}$ to get

$$n_a \sin\theta_0 = n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{4\epsilon_{b1}^2 \cos^2\theta_2'} \right]^{1/2}$$

$$\simeq n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2 \cos^2\theta_2'} \right]$$

small correction to Snell's law

for $\frac{\epsilon_{b2}}{\epsilon_{b1}} \ll 1$ can solve iteratively

to lowest order: $n_a \sin\theta_0 \simeq n_b \sin\theta_2'$

$$\Rightarrow \cos^2\theta_2' = 1 - \sin^2\theta_2' = 1 - \left(\frac{n_a}{n_b} \sin\theta_0\right)^2$$

so to next order

$$n_a \sin\theta_0 \simeq n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2 \left(1 - \frac{n_a^2}{n_b^2} \sin^2\theta_0\right)} \right]$$

or $$\sin\theta_2' \simeq \frac{n_a}{n_b} \sin\theta_0 \frac{1}{\left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{b1}^2 \left(1 - \frac{n_a^2}{n_b^2} \sin^2\theta_0\right)} \right]}$$

$$\leq \frac{n_a}{n_b} \sin\theta_0$$

result is that θ_2' is smaller than Snell's law would predict.

(2) for a good conductor, or absorbing region of a dielectric, $\epsilon_{b2} \gg \epsilon_{b1}$

to lowest order

$$n_a \sin\theta_0 = \sqrt{\mu_b \epsilon_{b1}} \left[\frac{1}{2} \frac{\epsilon_{b2}}{\epsilon_{b1} \cos\theta_2'} \right]^{1/2} \sin\theta_2'$$

$$n_a \sin\theta_0 = \sqrt{\frac{\mu_b \epsilon_{b2}}{2}} \frac{\sin\theta_2'}{\sqrt{\cos\theta_2'}}$$

very differnt from Snell's Law!

Snell's Law only holds if both media are transparent

$$\Rightarrow n_a^2 \sin^2\theta_0 = \frac{\mu_b \epsilon_{b2}}{2} \frac{\sin^2\theta_2'}{\cos\theta_2'} = \frac{\mu_b \epsilon_{b2}}{2} \frac{1 - \cos^2\theta_2'}{\cos\theta_2'}$$

$$\Rightarrow \cos^2\theta_2' + \left(\frac{2}{\mu_b \epsilon_{b2}}\right)(n_a^2 \sin^2\theta_0) \cos\theta_2' - 1 = 0$$

solve quadratic equation in $\cos\theta_2'$ to determine $\cos\theta_2'$. Then can use that in expressions for k_2' and k_2'' to determine those. We will have in this $\epsilon_{b2} \gg \epsilon_{b1}$ case that $k_1' \approx k_2'$

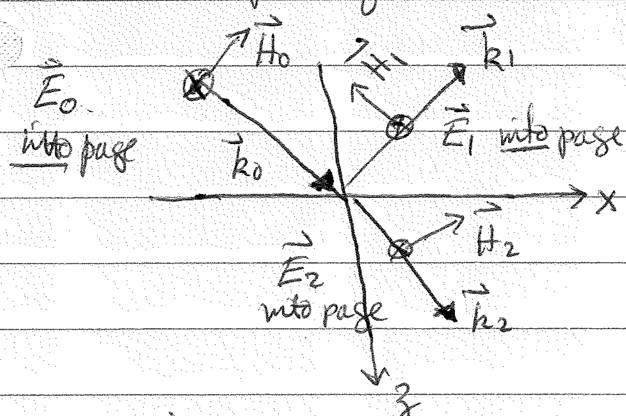
<u>Reflection coefficients</u>

Now we compute the <u>amplitude</u> of the reflected wave to determine how much of incident wave is reflected and how much is transmitted.

Consider two cases ① $\vec{E}_\omega$ is $\perp$ plane of incidence
② $\vec{E}_\omega$ lies in the plane of incidence

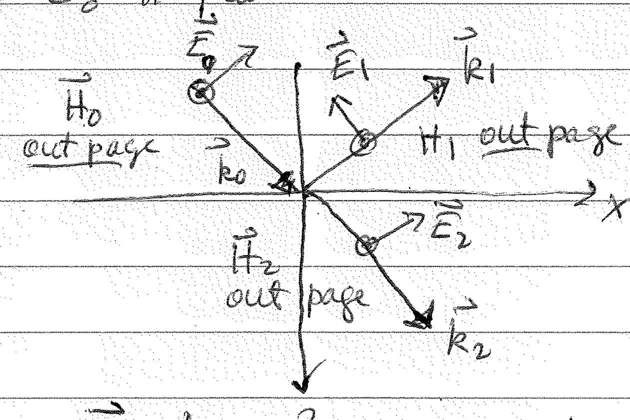
"plane of incidence" is the plane spanned by the wave vector $\vec{k}_0$ and the normal to the interface – in our case it is the xz plane

① $\vec{E}_0 \perp$ plane of incidence



$\Rightarrow$ $\vec{H}_0$ in plane of incidence
all $\vec{E}$'s are in $\hat{y}$ direction

② $\vec{E}_0 \parallel$ plane incidence



$\Rightarrow$ $\vec{E}_0$ in plane of incidence
all the $\vec{H}$'s are in y direction

<u>Continuity of y components</u>

i) $E_0 + E_1 = E_2$ | i) $H_0 + H_1 = H_2$

<u>continuity of x components</u>

$H_{0x} + H_{1x} = H_{2x}$ | $E_{0x} + E_{1x} = E_{2x}$

<u>Faraday</u>

$i\frac{\mu\omega}{c}\vec{H} = i\vec{k}\times\vec{E} \Rightarrow H_x = \frac{k_z c}{\omega\mu}E_y$

<u>Ampere</u>

$-i\frac{\omega\epsilon}{c}\vec{E} = i\vec{k}\times\vec{H} \Rightarrow E_x = \frac{-k_z c}{\omega\epsilon}H_y$

2) $\dfrac{k_{0z}}{\mu_a}(E_0 - E_1) = \dfrac{k_{2z}}{\mu_b} E_2$	2) $\dfrac{k_{0z}}{\epsilon_a}(H_0 - H_1) = \dfrac{k_{2z}}{\epsilon_b} H_2$
solve (1) and (2) for E_1 and E_2 in terms of E_0	solve (1) and (2) for H_1 and H_2 in terms of H_0
$E_1 = \dfrac{\mu_b k_{0z} - \mu_a k_{2z}}{\mu_b k_{0z} + \mu_a k_{2z}} E_0$	$H_1 = \dfrac{\epsilon_b k_{0z} - \epsilon_a k_{2z}}{\epsilon_b k_{0z} + \epsilon_a k_{2z}} H_0$
$E_2 = \dfrac{2\mu_b k_{0z}}{\mu_a k_{2z} + \mu_b k_{0z}} E_0$	$H_2 = \dfrac{2\epsilon_b k_{0z}}{\epsilon_a k_{2z} + \epsilon_b k_{0z}} H_0$

Define reflection coefficient in terms of the transported energy $R = \dfrac{|E_1|^2}{|E_0|^2} = \dfrac{|H_1|^2}{|H_0|^2}$

Reflection coefficients

① $\vec{E}_0 \perp$ plane incidence

$$R_\perp = \frac{|E_1|^2}{|E_0|^2} = \left| \frac{\mu_b k_{0z} - \mu_a k_{2z}}{\mu_b k_{0z} + \mu_a k_{2z}} \right|^2$$

② $\vec{E}_0 \parallel$ plane incidence

$$R_\parallel = \frac{|H_1|^2}{|H_0|^2} = \left| \frac{\epsilon_b k_{0z} - \epsilon_a k_{2z}}{\epsilon_b k_{0z} + \epsilon_a k_{2z}} \right|^2$$

Note: above are correct for an arbitrary medium B

i) Consider region of "total reflection"

$\Rightarrow$ $\text{Im}\,\epsilon_b = \epsilon_{b2} \approx 0$
$\text{Re}\,\epsilon_b = \epsilon_{b1} < 0$ $\Big\} \Rightarrow \vec{k}_2 = i\vec{K}_2$ where $\vec{K}_2$ is real

ie $\vec{k}_2$ pure imaginary

$$\Rightarrow \quad R_\perp = \left| \frac{\mu_b k_{0z} - i\mu_a K_{2z}}{\mu_b k_{0z} + i\mu_a K_{2z}} \right|^2$$

$$R_\parallel = \left| \frac{\epsilon_b k_{0z} - i\epsilon_a K_{2z}}{\epsilon_b k_{0z} + i\epsilon_a K_{2z}} \right|^2$$

both are of the form $\left|\frac{a-ib}{a+ib}\right|^2 = 1$ when a, b real

$\Rightarrow R_\perp = R_\parallel = 1$

Confirms that the material is completely reflecting

ii) Next consider when medium B is transparent

ϵ_b is real and $\epsilon_b > 0$

$$k_{0z} = \frac{\omega}{c}\sqrt{\mu_a \epsilon_a}\cos\theta_0 = \frac{\omega}{c} n_a \cos\theta_0$$
$$k_{2z} = \frac{\omega}{c}\sqrt{\mu_b \epsilon_b}\cos\theta_2 = \frac{\omega}{c} n_b \cos\theta_2$$

Snell's law holds so $n_a \sin\theta_0 = n_b \sin\theta_2$

Can write $R_\perp$ and $R_\parallel$ as functions of θ_0

for simplicity take $\mu_a = \mu_b = 1$

① $$R_\perp = \left(\frac{n_a \cos\theta_0 - n_b \cos\theta_2}{n_a \cos\theta_0 + n_b \cos\theta_2}\right)^2 = \left(\frac{\cos\theta_0 - \left(\frac{\sin\theta_0}{\sin\theta_2}\right)\cos\theta_2}{\cos\theta_0 + \left(\frac{\sin\theta_0}{\sin\theta_2}\right)\cos\theta_2}\right)^2$$

$$= \left(\frac{\sin\theta_2 \cos\theta_0 - \sin\theta_0 \cos\theta_2}{\sin\theta_2 \cos\theta_0 + \sin\theta_0 \cos\theta_2}\right)^2$$

$$R_\perp = \left(\frac{\sin(\theta_0 - \theta_2)}{\sin(\theta_0 + \theta_2)}\right)^2$$

for $\theta_0 = 0$, ie normal incidence, $\theta_2 = 0$

$$\Rightarrow R_\perp = \left(\frac{n_a - n_b}{n_a + n_b}\right)^2$$

if $n_a = n_b$, no reflection! (not surprising!)

② $$R_\parallel = \left(\frac{\epsilon_b n_a \cos\theta_0 - \epsilon_a n_b \cos\theta_2}{\epsilon_b n_a \cos\theta_0 + \epsilon_a n_b \cos\theta_2}\right)^2$$

use $\sqrt{\epsilon_b} = n_b$

$\sqrt{\epsilon_a} = n_a$

$$= \left(\frac{n_b \cos\theta_0 - n_a \cos\theta_2}{n_b \cos\theta_0 + n_a \cos\theta_2}\right)^2$$

$$= \left(\frac{\cos\theta_0 - \left(\frac{\sin\theta_2}{\sin\theta_0}\right)\cos\theta_2}{\cos\theta_0 + \left(\frac{\sin\theta_2}{\sin\theta_0}\right)\cos\theta_2}\right)^2$$

$$= \left(\frac{\sin\theta_0 \cos\theta_0 - \sin\theta_2 \cos\theta_2}{\sin\theta_0 \cos\theta_0 + \sin\theta_2 \cos\theta_2}\right)^2$$

$$R_\parallel = \left(\frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)}\right)^2$$

← after some algebra!

for $\theta_0 = 0$, then $\theta_2 = 0$

$$R_{\parallel} = \left(\frac{\epsilon_b n_a - \epsilon_a n_b}{\epsilon_b n_a + \epsilon_a n_b}\right)^2 = \left(\frac{n_b - n_a}{n_b + n_a}\right)^2 \quad \text{same as } R_{\perp}$$

So <u>for $\theta_0 = 0$</u>, $R_{\parallel} = R_{\perp}$ – this must be so since for $\theta_0 = 0$ there is no distinction between the $\perp$ and $\parallel$ cases

If $n_b = n_a$, $R_{\perp} = R_{\parallel} = 0$ no reflective wave

When $\theta_0 + \theta_2 = \pi/2$, then $\tan(\theta_0 + \theta_2) \to \infty$ and $R_{\parallel} = 0$

This occurs at an angle of incidence known as <u>Brewster's angle</u> θ_B, determined by

$$\underset{\substack{\uparrow \\ \theta_0}}{n_a \sin \theta_B} = n_b \sin\underset{\substack{\uparrow \\ \theta_2}}{\left(\frac{\pi}{2} - \theta_B\right)} = n_b \cos \theta_B$$

$$\Rightarrow \boxed{\tan \theta_B = \frac{n_b}{n_a}}$$

For incident wave at θ_B, reflected wave always has $\vec{E}_1 \perp$ plane of incidence, since $R_{\parallel} = 0$. If incoming wave has $\vec{E}_0 \parallel$ plane of incidence, then it gets completely transmitted. If $\vec{E}_0$ in general direction, reflected wave is <u>always</u> linearly polarized with $\vec{E}_1 \perp$ plane of incidence. – This is one method to create polarized light wave.