Kramers - Kronig Relation

We saw that $\vec{P}_\omega = \alpha(\omega)\vec{E}_\omega \Rightarrow \vec{P}(t) = \int_{-\infty}^{\infty} dt'\, \tilde{\alpha}(t-t')\vec{E}(t')$

<u>Causal</u> response ie $\tilde{\alpha}(t) = 0$ for $t < 0$

$\Rightarrow$ $\alpha(\omega)$ has no poles in upper half of complex ω plane (UHP)

For any complex $\bar{\omega}$ in upper half of complex ω plane,

$$\alpha(\bar{\omega}) = \frac{1}{2\pi i}\oint \frac{\alpha(\omega')}{\omega'-\bar{\omega}}\,d\omega'$$

since no poles of α in UHP

contour along real axis, closed at infinity in UHP. The closing semicircle at infinity gives no contribution assuming $\alpha(\omega)$ decays quickly enough as $|\omega| \to \infty$

only pole of integrand is at $\omega' = \bar{\omega}$

$$\Rightarrow \quad \alpha(\bar{\omega}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega'\, \frac{\alpha(\omega')}{\omega'-\bar{\omega}}$$

Now consider $\bar{\omega} = \omega + i\delta$ where ω and δ are real and $\delta \to 0$

$$\alpha(\omega) = \lim_{\delta\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega'\, \frac{\alpha(\omega')}{\omega'-\omega-i\delta}$$

Now $$\frac{1}{\omega'-\omega-i\delta} = P\left(\frac{1}{\omega'-\omega}\right) + i\pi\delta(\omega'-\omega)$$

principle part

$$\Rightarrow \alpha(\omega) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{\alpha(\omega')\,d\omega'}{\omega'-\omega}$$

$$\Rightarrow \boxed{\begin{aligned} \mathrm{Re}\,\alpha(\omega) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Im}\,\alpha(\omega')\,d\omega'}{\omega'-\omega} \\ \mathrm{Im}\,\alpha(\omega) &= -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Re}\,\alpha(\omega')\,d\omega'}{\omega'-\omega} \end{aligned}}$$

Kramer Kronig relations

If know Re α or Im α can reconstruct full complex α

True for any causal response function

details:

$$\alpha(\omega) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega'-\omega} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega'\, i\pi\delta(\omega'-\omega)\,\alpha(\omega)$$

$$= \frac{1}{2\pi i} P \int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega'-\omega} + \frac{1}{2}\alpha(\omega)$$

$$\Rightarrow \frac{1}{2}\alpha(\omega) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega'-\omega}$$

or $\alpha(\omega) = \dfrac{1}{\pi i} \displaystyle\int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega'-\omega}$ as above

Radiation From Moving Charges

In Lorentz gauge $\frac{1}{c}\frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -4\pi \rho$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{j}$$

wave equation with source

$$\vec{B} = \vec{\nabla}\times\vec{A}$$
$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

If we can solve wave equation with source (inhomogeneous wave equation) then we are in principle done! To do this we want to find the <u>Green's function</u> for the wave equation

Recall <u>from statics</u>: $\nabla^2 \phi = -4\pi\rho$

Greens function satisfies $\nabla^2 G(\vec{r}) = -4\pi\,\delta(\vec{r})$

then $\phi(\vec{r}) = \int d^3r'\, G(\vec{r}-\vec{r}')\,\rho(\vec{r}') + \phi_0$

solution for infinite volume that vanishes as $r\to\infty$ is

$$G(\vec{r}-\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} \qquad \nabla^2\phi_0 = 0$$

For <u>wave equation</u> we want solution to

$$\nabla^2 G(\vec{r},t;\vec{r}',t') - \frac{1}{c^2}\frac{\partial^2 G(\vec{r}t;\vec{r}'t')}{\partial t^2} = -4\pi\,\delta(\vec{r}-\vec{r}')\,\delta(t-t')$$

then we will have

$$\phi(\vec{r},t) = \int dt' \int d^3r'\, G(\vec{r},t;\vec{r}'t')\,\rho(\vec{r}',t') + \phi_0$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int dt' \int d^3r'\, G(\vec{r}t;\vec{r}'t)\,\vec{j}(\vec{r}',t') + \vec{A}_0$$

where $\nabla^2 \Phi_0 - \frac{1}{c^2}\frac{\partial^2 \Phi_0}{\partial t^2} = 0$ similarly for $\vec{A}_0$

Φ_0 and $\vec{A}_0$ could describe an incoming wave for example

To construct the Green's function.

For infinite space (but not, for example, inside a cavity)
$G(\vec{r}t; \vec{r}'t') = G(\vec{r}-\vec{r}', t-t')$
express as Fourier transform

$$G(\vec{r}, t) = \int \frac{d^3k\, d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r}, t) = \int \frac{d^3k\, d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$'' \qquad = \int \frac{d^3k\, d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) \left[-k^2 + \frac{\omega^2}{c^2}\right] e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$-4\pi\delta(\vec{r})\,\delta(t) = -4\pi \int \frac{d^3k\, d\omega}{(2\pi)^4} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

equate Fourier amplitudes

$$\Rightarrow \quad \left[-k^2 + \frac{\omega^2}{c^2}\right] \tilde{G}(\vec{k}, \omega) = -4\pi$$

$$\boxed{\tilde{G}(\vec{k}, \omega) = \frac{4\pi c^2}{k^2c^2 - \omega^2}} \qquad \text{when } \omega^2 \neq c^2k^2$$

$$G(\vec{r}, t) = \int \frac{d^3k\, d\omega}{(2\pi)^4} \frac{4\pi c^2}{k^2c^2 - \omega^2} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

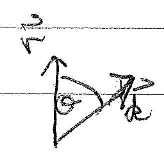
poles at $\omega = \pm ck$

In evaluating the ω integral we have to know how to treat the poles on the real axis so that $G(\vec{r},t)$ will have the desired behavior.

What we want is for $G(\vec{r},t)$ to be <u>causal</u>, ie $G(\vec{r},t)=0$ for $t<0$, so $\Phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ depend only on the values of the sources at <u>earlier</u> times $t'<t$.

$$\int d^3k\, e^{i\vec{k}\cdot\vec{r}}\,\tilde{G}(\vec{k},\omega) = 2\pi\int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk\, k^2\, e^{ikr\cos\theta}\,\tilde{G}(k,\omega)$$

$\vec{k}=(k,\theta,\varphi)$ in spherical coordinates



$\mu = \cos\theta$
$d\mu = -d\theta\sin\theta$

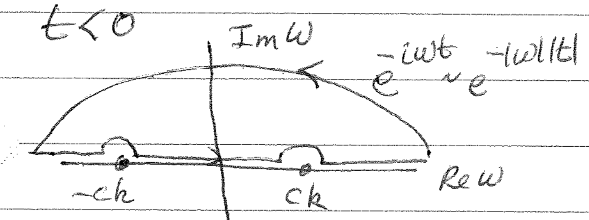
$$= 2\pi\int_{-1}^{1} d\mu \int_0^{\infty} dk\, k^2\, e^{ikr\mu}\,\tilde{G}(k,\omega)$$

$$= 4\pi\int_0^{\infty} dk\, k^2\, \frac{\sin kr}{kr}\,\tilde{G}(k,\omega)$$

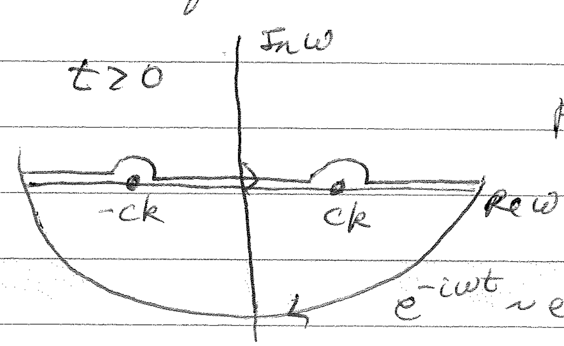
$$G(\vec{r},t) = -\frac{c^2}{\pi^2}\int_0^{\infty} dk\, k^2\,\frac{\sin kr}{kr}\int_C \frac{e^{-i\omega t}}{(\omega+ck)(\omega-ck)}\,d\omega$$

contour along real axis, but deformed to go around the poles

for $t<0$, $e^{-i\omega t}$ will decay exponentially fast for large $|\omega|$ in the <u>upper half complex</u> (UHP) ω plane $\Rightarrow$ can close contour in UHP for $t<0$. If we want $G=0$ for $t<0$, there should therefore be no poles in UHP. The contour C we want is therefore:



no poles in contour $\Rightarrow G=0$



poles within contour $\Rightarrow G\neq 0$

with this convention for the contour C we can evaluate the ω-integral using Cauchy's residue theorem

For $t>0$, $$\int \frac{e^{-i\omega t}\,d\omega}{(\omega+ck)(\omega-ck)} = -2\pi i\left[\frac{e^{-ickt}}{2ck} - \frac{e^{ickt}}{2ck}\right]$$

$$= -\frac{2\pi \sin(ckt)}{ck}$$

$$G(\vec{r},t) = \frac{2c}{\pi r}\int_0^\infty dk\, \sin(kr)\sin(ckt) = \frac{c}{\pi r}\int_{-\infty}^{\infty} dk\, \frac{(e^{ikr}-e^{-ikr})(e^{ickt}-e^{-ickt})}{(-4)}$$

$$= -\frac{c}{2r}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left\{e^{i(r+ct)k} + e^{-i(r+ct)k} - e^{i(r-ct)k} - e^{-i(r-ct)k}\right\}$$

each integral would give a δ-function, but for 1st two terms $\delta(r+ct)=0$ since here $t>0$ (by definition) and $r=|\vec{r}|\geq 0$ so the arguement will never vanish.

$$\boxed{G(\vec{r},t) = \frac{c}{r}\,\delta(r-ct) = \frac{\delta(t-r/c)}{r}}$$ using $\delta(ax) = \frac{1}{a}\delta(x)$

$$G(\vec{r}t,\vec{r}'t') = \begin{cases} \dfrac{\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} & t-t'>0 \\ 0 & t-t'<0 \end{cases}$$

Green's function for wave equation in free space

$G \neq 0$ only on "light cone" that ~~connects~~ emanates from (r',t'), ie when $|\vec{r}-\vec{r}'| = c(t-t')$.
Signal from source at (r',t') travels with c

$$\phi(\vec{r},t) = \phi_0(\vec{r},t) + \int d^3r' \int_{-\infty}^{t} dt' \frac{\delta\left(t - t' - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} \rho(\vec{r}',t')$$

$$\vec{A}(\vec{r},t) = \vec{A}_0(\vec{r},t) + \int \frac{d^3r'}{c} \int_{-\infty}^{t} dt' \frac{\delta\left(t - t' - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} \vec{j}(\vec{r}',t')$$

Apply to a single moving point charge

$$\rho(\vec{r},t) = q\,\delta(\vec{r} - \vec{r}_0(t))$$
$$\vec{j}(\vec{r},t) = q\,\vec{v}(t)\,\delta(\vec{r} - \vec{r}_0(t)) \qquad \text{where } \vec{v}(t) = \frac{d\vec{r}_0}{dt}$$

Then

$$\phi(\vec{r},t) = q \int dt' \frac{\delta\left(t - t' - \frac{1}{c}|\vec{r} - \vec{r}_0(t')|\right)}{|\vec{r} - \vec{r}_0(t')|}$$

because of the $\vec{r}_0(t')$ in the argument of the $\delta()$ function the t' dependence is not of the simple form $t' - t_0$.

We can write

$$g(t') \equiv t' + \frac{1}{c}|\vec{r} - \vec{r}_0(t')|$$

then

$$\phi(\vec{r},t) = q \int dt' \frac{\delta(t - g(t'))}{|\vec{r} - \vec{r}_0(t')|}$$

$$= q \int \frac{\delta(t - g(t'))}{|\vec{r} - \vec{r}_0(t')|}\, dg \left(\frac{dt'}{dg}\right)$$

$$= \frac{q}{|\vec{r} - \vec{r}_0(t')|} \frac{1}{(dg/dt')} \Bigg|_{t' \text{ such that } g(t') = t}$$