

$$\vec{\nabla} \times \vec{B} = k_5 \vec{\nabla} \left[\int d^3r' \vec{\nabla} \cdot \left(\frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right) \right]$$

$$- k_5 \int d^3r' \vec{j}(\vec{r}') \nabla^2 \left(\frac{1}{|\vec{r}-\vec{r}'|} \right)$$

in the 2nd term, $\nabla^2 \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = -4\pi \delta(|\vec{r}-\vec{r}'|)$

in the 1st term, $\vec{\nabla} \cdot \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \vec{j}(\vec{r}') \cdot \vec{\nabla} \left(\frac{1}{|\vec{r}-\vec{r}'|} \right)$

$$= -\vec{j}(\vec{r}') \cdot \vec{\nabla}' \left(\frac{1}{|\vec{r}-\vec{r}'|} \right)$$

since $\vec{\nabla} = -\vec{\nabla}'$

So $\int d^3r' \vec{\nabla} \cdot \left(\frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right) = - \int d^3r' \vec{j}(\vec{r}') \cdot \vec{\nabla}' \left(\frac{1}{|\vec{r}-\vec{r}'|} \right)$

integrate by parts $= \int d^3r' (\vec{\nabla}' \cdot \vec{j}(\vec{r}')) \left(\frac{1}{|\vec{r}-\vec{r}'|} \right)$

surface term $\rightarrow 0$ as

we take surface $\rightarrow \infty$

since $\vec{j} \rightarrow 0$ as $r \rightarrow \infty$

But for magnetostatics $\vec{\nabla} \cdot \vec{j} = 0 \Rightarrow$ only 2nd term remains

Thus, for magnetostatics

$$\vec{\nabla} \times \vec{B} = 4\pi k_5 \vec{j} \quad \text{Ampere's law}$$

integral form $\oint_C d\vec{l} \cdot \vec{B} = 4\pi k_5 \int_S da \hat{n} \cdot \vec{j}$

\uparrow curve \downarrow surface
 \uparrow curve bounding surface

Although above diff eqs were derived starting from a "non-relativistic"

point-charge Biot-Savart law, they actually remain true for all magnetostatic situations

(15)

So far electrostatics $\begin{cases} \vec{\nabla} \cdot \vec{E} = 4\pi k_1 \rho & \text{Gauss} \\ \vec{\nabla} \times \vec{E} = 0 \end{cases}$

magnetostatics $\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = 4\pi k_5 \vec{j} & \text{Ampere} \end{cases}$

charge conservation $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$

Time Dependent situations

Faraday's law of induction $\vec{\nabla} \times \vec{E} \neq 0!$

$$\oint_C \vec{dl} \cdot \vec{E} = -k_3 \frac{\partial}{\partial t} \int_S da \hat{n} \cdot \vec{B}$$

emf around loop $\mathcal{E}_{\text{mf}} = -k_3 \frac{d\Phi}{dt}$ mag flux

choose \hat{n} voltage around closed loop \sim -time rate of change of magnetic flux through loop
 and $d\vec{l}$ consistent with Right Hand Rule

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -k_3 \frac{\partial \vec{B}}{\partial t}}$$

k_3 is universal constant

Maxwell correction to Ampere's law

In our derivation of $\vec{\nabla} \times \vec{B} = 4\pi k_5 \vec{j}$ we used $\vec{\nabla} \cdot \vec{j} = 0$. This is only true for magnetostatics - it is NOT true in general

Alternatively, since $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$ always, if Ampere's law was true, we would necessarily conclude that $\vec{\nabla} \cdot \vec{j} = 0$. But $\vec{\nabla} \cdot \vec{j} = -\frac{\partial \rho}{\partial t} \neq 0$ in general.

Proposed correction: $\vec{\nabla} \times \vec{B} = 4\pi k_5 \vec{j} + \vec{W}$

where \vec{W} must be such that charge conservation holds.

Now

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0 = 4\pi k_5 \vec{\nabla} \cdot \vec{j} + \vec{\nabla} \cdot \vec{W}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{W} = -4\pi k_5 \vec{\nabla} \cdot \vec{j} = 4\pi k_5 \frac{\partial \rho}{\partial t} \quad \text{by charge conserv}$$

$$= \frac{k_5}{k_1} \frac{\partial \vec{\nabla} \cdot \vec{E}}{\partial t} \quad \text{by Gauss Law}$$

$$\Rightarrow \vec{W} = \frac{k_5}{k_1} \frac{\partial \vec{E}}{\partial t}$$

So corrected Ampere's Law is

$$\boxed{\vec{\nabla} \times \vec{B} = 4\pi k_5 \vec{j} + \frac{k_5}{k_1} \frac{\partial \vec{E}}{\partial t}}$$

EM waves

$$\begin{aligned} \text{Now consider } \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) &= \vec{\nabla} (\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} \\ &= -\nabla^2 \vec{B} \quad \text{as } \vec{\nabla} \cdot \vec{B} = 0 \end{aligned}$$

If there are no sources ($\rho = 0, \vec{j} = 0$) then

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\nabla^2 \vec{B} = \frac{k_5}{k_1} \frac{\partial \vec{\nabla} \times \vec{E}}{\partial t}$$

$$= -\frac{k_5 k_3}{k_1} \frac{\partial^2 \vec{B}}{\partial t^2} \quad \text{by Faraday}$$

$$\nabla^2 \vec{B} = \frac{k_5 k_3}{k_1} \frac{\partial^2 \vec{B}}{\partial t^2} \quad \text{this is the wave equation}$$

$$\Rightarrow \frac{k_5 k_3}{k_1} \text{ has units of (velocity)}^{-2}$$

Since we know that the above wave equation describes electromagnetic waves, i.e. light, then

$$\frac{k_5 k_3}{k_1} = \frac{1}{c^2}$$

we already had $\frac{k_4 k_5}{k_1} = \frac{1}{c^2}$

$$\Rightarrow k_3 = k_4$$

$\Rightarrow k_1$ and k_4 are arbitrary - they can be chosen to be anything by adjusting the units of g and B . k_3 and k_5 are then fixed by $\frac{k_4 k_5}{k_1} = \frac{1}{c^2}$) $k_3 = k_4$

Popular systems of E & M units

	k_1	$k_3 = k_4$	k_5	
MKS or SI	$\frac{1}{4\pi\epsilon_0}$	1	$\frac{\mu_0}{4\pi}$	$(\epsilon_0 \mu_0 = \frac{1}{c^2})$
Gaussian or CGS	1	$\frac{1}{c}$	$\frac{1}{c}$	
Rationalized Gaussian	$\frac{1}{4\pi}$	$\frac{1}{c}$	$\frac{1}{4\pi c}$	

In MKS, charges are measured in "coulombs"

current measured in "amps"

magnetic field measured in "tesla" = "weber/m²"

In CGS, charges are measured in "statcoulombs"

current measured in "statamperes"

magnetic field measured in "gauss"

1 tesla = 10⁴ gauss

We will use the CGS or Gaussian units

1) $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$

3) $\vec{\nabla} \cdot \vec{B} = 0$

2) $\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$

4) $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$

Maxwell's Eqs !!

$\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$

Lorentz force

Physical content of Maxwell's eqs.

1) Gauss Law for electric field - charge is source of \vec{E} field. Field lines can begin and end at point charges

2) Faraday's Law of induction - time varying magnetic flux produces circulating \vec{E} -field

3) Gauss Law for magnetic fields - no magnetic monopoles. Magnetic field lines are continuous, they either close upon themselves or go off to infinity, they cannot begin nor end at any point.

4) Ampere's Law + Maxwell's correction - electric current is a source for circulating \vec{B} -field; so is a time varying \vec{E} -field. Maxwell's correction is necessary to have charge conservation and to give electromagnetic waves.

Note:

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{E}) = 0$ as div of curl always vanishes
then (2) $\Rightarrow -\frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{B}) = 0$

So if $\vec{\nabla} \cdot \vec{B} = 0$ at $t=0$, Equ (2) requires that $\vec{\nabla} \cdot \vec{B}$ remains zero for all time

Similarly

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$
then (4) $\Rightarrow \frac{4\pi}{c} \vec{\nabla} \cdot \vec{j} + \frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E}) = 0$

use charge continuity $\vec{\nabla} \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$ to get

$$-\frac{4\pi}{c} \frac{\partial \rho}{\partial t} + \frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E}) = 0$$

$$\Rightarrow \frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E} - 4\pi \rho) = 0$$

So if $\vec{\nabla} \cdot \vec{E} = 4\pi \rho$ at $t=0$, Equ (4) requires that $\vec{\nabla} \cdot \vec{E} = 4\pi \rho$ for all time

Thus Eqs (1) and (3) can be viewed as "initial conditions". If they are true for any particular moment in time, then Eqs (2) and (4) ensure that they remain true for all time.

Eqs (2) and (3) are also referred to as the homogeneous Maxwell Eqs - they involve only the fields \vec{E} and \vec{B} and not the sources ρ and \vec{j} . Eqs (1) and (4) are referred to as the inhomogeneous Maxwell's Eqs - they involve the sources ρ and \vec{j} .

Electromagnetic Potentials & Gauge Invariance

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi\vec{j}}{c} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

Consider first statics.

Electrostatics

$$\vec{\nabla} \times \vec{E} = 0 \quad \text{since } \frac{\partial \vec{B}}{\partial t} = 0 \text{ for statics}$$

From vector calculus we know that if the curl of a vector is ~~also~~ everywhere zero, then we can always write that vector field as the gradient of some scalar function ϕ

$$\vec{E} = -\vec{\nabla}\phi \Rightarrow \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times (\vec{\nabla}\phi) = 0$$

ϕ is electrostatic potential

Gauss Law becomes

$$\vec{\nabla} \cdot \vec{E} = -\vec{\nabla} \cdot (\vec{\nabla}\phi) = -\nabla^2\phi = 4\pi\rho$$

$$\boxed{\nabla^2\phi = -4\pi\rho} \quad \text{Poisson's Equation}$$

In regions where $\rho = 0$, we have

$$\nabla^2\phi = 0 \quad \text{Laplace's Equation}$$

In our discussion of Coulomb's Law we saw that the electric field from a distribution of localized charges was

$$\begin{aligned}\vec{E}(\vec{r}) &= \int d^3r' \rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \\ &= -\vec{\nabla} \left[\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \right] = -\vec{\nabla}\phi\end{aligned}$$

We therefore see that the solution to Poisson's equ for a localized charge distribution ρ (with $\vec{E} = 0$ as $\vec{r} \rightarrow \infty$) is

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

We will soon spend a fair amount of time learning new ways to solve $\nabla^2\phi = \rho$, both for arbitrary ρ where we want an approx to the above integral (multipole expansion), and for cases where ϕ or $\vec{\nabla}\phi$ are predetermined on the surfaces of specified regions of space, such as conducting surfaces (boundary value problems).

Magnetostatics

$$\vec{\nabla} \cdot \vec{B} = 0$$

From vector calculus we know that if the divergence of a vector field vanishes everywhere, then it can always be written as the curl of another vector field \vec{A}

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

\vec{A} is the magnetic vector potential

This remains true in general - not just in magneto statics

Ampere's law becomes

$$\vec{\nabla} \times \vec{B} = \frac{4\pi \vec{j}}{c} \quad (\text{in magnetostatics } \frac{\partial \vec{E}}{\partial t} = 0)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{4\pi \vec{j}}{c}$$

$$\Rightarrow \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \frac{4\pi \vec{j}}{c}$$

magnetostatic gauge invariance

There are many possible vector potentials \vec{A} that result in the same \vec{B} . If \vec{A} is such that $\vec{\nabla} \times \vec{A} = \vec{B}$, then $\vec{A}' = \vec{A} + \vec{\nabla} \chi$ also gives $\vec{\nabla} \times \vec{A}' = \vec{B}$, since $\vec{\nabla} \times \vec{\nabla} \chi = 0$ for any scalar function $\chi(r)$.

Therefore we can always choose to represent \vec{B} by a vector potential \vec{A} such that $\vec{\nabla} \cdot \vec{A} = 0$.

proof: Suppose we had $\vec{B} = \vec{\nabla} \times \vec{A}$ for some \vec{A} with $\vec{\nabla} \cdot \vec{A} = D(\vec{r}) \neq 0$. Construct an $\vec{A}' = \vec{A} + \vec{\nabla} \chi$ with χ chosen as follows:

$$\vec{\nabla} \cdot \vec{A}' = \vec{\nabla} \cdot \vec{A} + \nabla^2 \chi = 0 \Rightarrow \nabla^2 \chi = -\vec{\nabla} \cdot \vec{A} = D$$

Solve for χ , for example

$$\chi(\vec{r}) = \frac{\int d^3 r' D(\vec{r}')}{4\pi |\vec{r} - \vec{r}'|}$$

we thus have constructed an \vec{A}' such that $\vec{\nabla} \times \vec{A}' = \vec{B}$ and $\vec{\nabla} \cdot \vec{A}' = 0$.

This freedom to choose various \vec{A} 's that give the same \vec{B} is called gauge invariance. Imposing a particular additional constraint on \vec{A} that removes this freedom is called fixing the gauge. The choice $\vec{\nabla} \cdot \vec{A} = 0$ is usually known as the Coulomb gauge (or sometimes the Landau gauge). Going from \vec{A} to $\vec{A}' = \vec{A} + \vec{\nabla} \chi$ is called making a gauge transformation.

"Working in the Coulomb gauge" with $\vec{\nabla} \cdot \vec{A} = 0$, Ampere's Law becomes

$$\boxed{\nabla^2 \vec{A} = -\frac{4\pi}{c} \vec{j}} \quad \text{Poisson's Equ.}$$

For a localized current density

$$\vec{A}(\vec{r}) = \frac{1}{c} \int d^3 r' \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$